$R > 0$

Example 3: For the radius R circle traversed counterclockwise, $\gamma(t) = R\,e^{it}$, $0 \le t \le 2\pi$, compute

$$\cancel{\int_\gamma \frac{1}{z}\,dz}.$$

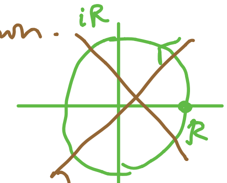
$$\int \frac{1}{z - z_0}\,dz$$

Friday in brown.

$\gamma(t) = z_0 + Re^{it}$

$0 \le t \le 2\pi$

($R > 0$ rad.)



$\gamma'(t) = Rie^{it}$

come back to this.

$$\int_0^{2\pi} \underbrace{\frac{1}{\cancel{z_0} + Re^{it} - \cancel{z_0}}}_{f(\gamma(t))} \underbrace{iRe^{it}}_{\gamma'(t)}\,dt$$

$$= \int_0^{2\pi} i\,dt = \boxed{2\pi i}$$

B3 Integral estimate: Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$\le \int_a^b |f(\gamma(t))\gamma'(t)|\,dt \qquad (A3)$$

$$= \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

Def Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\int_\gamma |f(z)|\,|dz| := \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

Using the definition, we see that the shorthand for the integral estimate in *B3* is

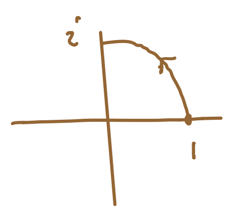
$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|.$$

Note that $|dz| = |\gamma'(t)|\,dt$ is the element of arclength.

Example 4: In Example 2, you showed that for $\gamma(t) = e^{it}$, $0 \le t \le \dfrac{\pi}{2}$,

$$\int_\gamma z\,dz = -1.$$

$|-1| \le \pi/2$ ✓



Compute

$$\int_\gamma |z|\,|dz| = \pi/2$$

and verify the integral estimate *B3*.

$z = e^{it}$

$dz = ie^{it}dt$

$$\int_0^{\pi/2} \underbrace{|e^{it}|}_{1}\,\underbrace{|ie^{it}|}_{1}\,dt = \pi/2$$

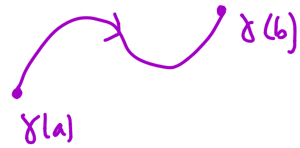
$|z| = 1$ on arc.

$|dz|$ = arclength

$\int_\gamma |z||dz| = 1 \cdot \text{length} = 1\,\frac{\pi}{2}.$

B3 Theorem FTC for complex line integrals Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ continuous, $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. If f has an analytic antiderivative in A, i.e. $F' = f$, then complex line integrals only depend on the endpoints of the curve γ, via the formula

$$\int_\gamma f(z)\, \mathrm{d}z := F(\gamma(b)) - F(\gamma(a))$$



proof:

$$\| $$

$$\int_a^b \underbrace{f(\gamma(t))\,\gamma'(t)}_{F'(\gamma(t))\,\gamma'(t)}\, dt$$

chain rule for curves

$$\int_a^b \frac{d}{dt} F(\gamma(t))\, dt \overset{!}{=} F(\gamma(t))\Big]_a^b \quad \blacksquare$$

part a. (A4)

Example 5: Redo Example 2 using the FTC: $\gamma(t) = e^{it}$, $0 \le t \le \frac{\pi}{2}$, $f(z) = z$,

$$\int_\gamma z\, \mathrm{d}z = \frac{z^2}{2}\Big]_{\gamma(0)}^{\gamma(\pi/2)} = \frac{z^2}{2}\Big]_{1 = e^{i0}}^{i = e^{i\pi/2}} = \frac{i^2}{2} - \frac{1}{2} = -\frac{1}{2} - \frac{1}{2} = -1$$

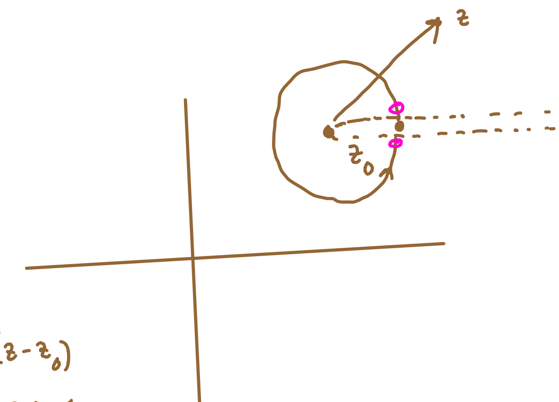
$f(z) = z$

$F(z) = \frac{z^2}{2}$

Example 6: Could you redo example 3 with the FTC? For the radius R circle traversed counterclockwise, $\gamma(t) = R\,e^{it}$, $0 \le t \le 2\pi$, compute

$$\cancel{\int_\gamma \frac{1}{z}\, \mathrm{d}z}.$$

$\gamma(z) = z_0 + Re^{it}$



$$\int_\gamma \frac{1}{z - z_0}\, dz$$

$F(z) = \log(z - z_0)$

$= \ln|z - z_0| + i \arg(z - z_0)$

$0 < \arg(z - z_0) < 2\pi$

$$\| $$

$$\log(z - z_0)\Big]_{(z_0+R)^+}^{(z_0+R)^-}$$ ← started just above cut, took limit

$$\| $$

$$\cancel{\ln R} + i\,2\pi - (\cancel{\ln R} - 0i) = 2\pi i \quad ✓$$

The connection between contour integrals and Calc 3 line integrals:

Let $A \subseteq \mathbb{C}$ open, $f: A \rightarrow \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{C}$ a C^1 curve. write

$$\gamma(t) = x(t) + i\,y(t),$$
$$f(z) = u(x, y) + i\,v(x, y).$$

Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

$$= \int_a^b (u(x(t), y(t)) + i\,v(x(t), y(t)))(x'(t) + i\,y'(t))\,\mathrm{d}t$$

$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)dt \; + i\int_a^b v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)dt$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \quad \checkmark$$

On Friday we'll combine this Calc 3 line integral way of writing contour integrals with Calc 3 Green's Theorem, for some interesting results.

Math 4200
Friday September 20

2.1-2.2 Contour integrals and Green's Theorem. We'll finish the examples and discussion related to contour integrals in Wednesday's notes first, before proceding into today's. One focus today is that if an analytic function has an antiderivative, then the value of a contour integral only depends on the starting and ending points of the contour (FTC for contour integrals). On Monday we'll turn that discussion around to use contour integrals in simply connected domains, to *find* the antiderivatives of analytic functions.

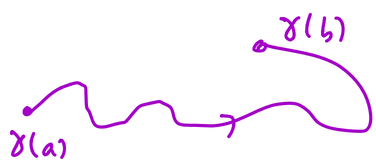
Announcements: We need your Wed notes...

Warm up exercise

For $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve and $f : \mathbb{C} \to \mathbb{C}$ continuous, how do you compute

$z = \gamma(t)$
$dz = \gamma'(t)\,dt$

$$\int_\gamma f(z)\,\mathrm{d}z \ ? \quad = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$



If $\boxed{f(z) = F'(z)}$ $\forall\, z$ in our domain, what is the shortcut for computing the contour integral above?

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

just depends on the terminal & initial pts of γ (& on F)

Recall *Green's Theorem* for real line integrals from multivariable Calculus, for C^1 vector fields $[P(x,y), Q(x,y)]$ around oriented boundaries of planar domains. A. (If you're rusty about why it's true there's a Wikipedia page; I also added an appendix page reminder from some old class notes, at the end of today's notes.)

I forgot

$$\int_{\delta A} P\,dx + Q\,dy = \iint_A Q_x - P_y\,dA$$

∂A ↑ bdry of A

CR: $u_x = v_y$
$-u_y = v_x$

What does Green's Theorem say about $\int_{\delta A} f(z)\,dz$ if f is C^1 and analytic on the closure $\overline{A}$? Hint: Cauchy-Riemann. (Note, we are interpreting the boundary integral as a sum of contour integrals if the boundary has more than one component.)

$$\oint_{\partial A} (u+iv)(dx+i\,dy) = \oint_{\partial A} \underbrace{u}_{P}\,dx \underbrace{-v}_{Q}\,dy + i\oint_{\partial A} \underbrace{v}_{P}\,dx + \underbrace{u}_{Q}\,dy$$

$$= \iint_A \underbrace{-v_x - u_y}_{0 \text{ CR}}\,dA + i\iint_A \underbrace{u_x - v_y}_{0 \text{ CR.}}\,dA$$

$Q_x - P_y$ $\quad$ $Q_x - P_y$

$$\int_{\partial A} f(z)\,dz = 0\,!!$$

↑ a sum of 4 contour integrals

Example. Can you compute

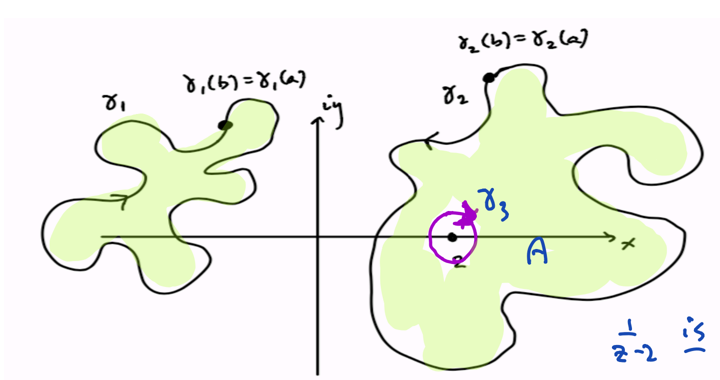
$$\int_{\gamma_1} \frac{1}{z-2}\,dz \qquad\qquad \int_{\gamma_2} \frac{1}{z-2}\,dz\ ?$$

(You can use partial fractions and these tricks to compute the contour integral of any rational function, around any simple closed curve.)

$$\int_{\gamma_1} \frac{1}{z-2}\,dz = 0$$

f analytic inside γ_1

$$\int_{\gamma_2} \frac{1}{z-2}\,dz$$



$\frac{1}{z-2}$ is analytic in $A \setminus D(2;R)$
R small.

Green's Thm

$$\Rightarrow \int_{\gamma_2} \frac{1}{z-2}\,dz + \underbrace{\int_{\gamma_3} \frac{1}{z-2}\,dz}_{-2\pi i} = \iint_{A\setminus D} 0 = 0$$

!!

$$\boxed{\Rightarrow \int_{\gamma_2} \frac{1}{z-2}\,dz = 2\pi i}$$

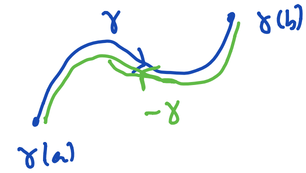
Contour curve algebra:

Def Let $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ be a C^1 curve, with range $\gamma([a, b]) \subseteq A$ an open set. Then we define the curve $-\gamma : [a, b] \to A$ by

"$-\gamma$"

$$-\gamma(t) := \gamma((a + b - t)),$$

@ $t = a$: $\gamma(b)$
$t = b$: $\gamma(a)$



i.e. traversing the curve in the opposite direction.

Note, by our discussion of contour integrals in terms of Riemann sums yesterday, if f is a C^1 analytic function on A, then

$$\int_{-\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz .$$

It's probably also worth verifying this with the chain rule, just for practice: no time today ☹

Now, consider piecewise C^1 contours that piece together continuously:

<u>Def</u>: Let $\gamma_j : [a_j, b_j] \to \mathbb{C}$ be $C^1, j = 1, 2 \ldots, n$. Require $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}), j = 1, \ldots n-1$

$$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}), \qquad j = 1, \ldots n-1.$$

Then $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_n]$ will be our notation for the piecewise C^1 path obtained from following the paths in order. (The text acually requires that the intervals $[a_j, b_j]$ piece together as well, i.e. $b_j = a_{j+1}$, which we could always accomplish by reparameterizing the curves if necessary. And in that case γ would *actually* be a piecewise C^1 function on the amalgamated interval $[a_1, b_n]$.)

<u>Def</u> For γ as above, define $\gamma_1(a_1)$ to be the *initial point* of γ, and $\gamma_n(b_n)$ to be the *terminal point.*

<u>Def</u> If $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_n]$ is piecewise C^1 as above we write

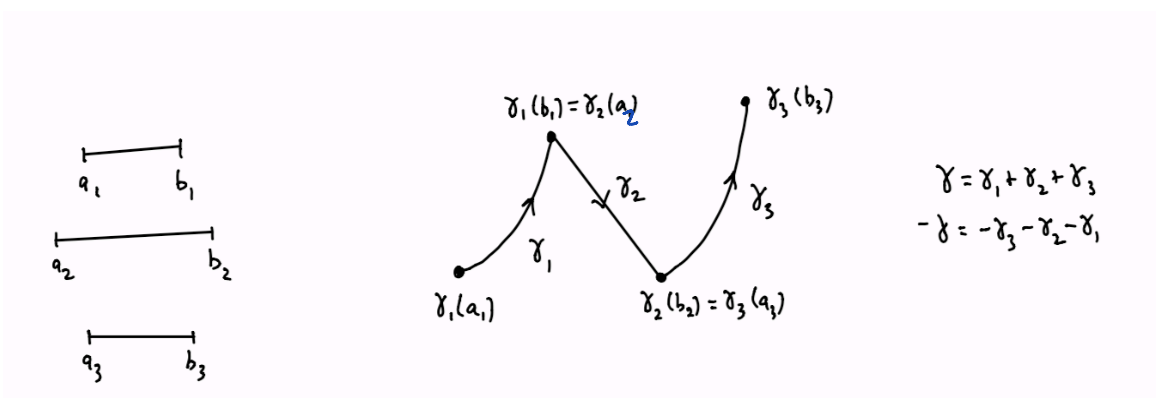
$$\gamma := \gamma_1 + \gamma_2 + \ldots + \gamma_n$$

and define the contour $-\gamma$ by $-\gamma = [-\gamma_n, -\gamma_{n-1}, \ldots, -\gamma_2, -\gamma_1]$ so

$$-\gamma := -\gamma_n - \gamma_{n-1} - \ldots - \gamma_2 - \gamma_1.$$

And we define the contour integral

$$\int_\gamma f(z)\, dz = \int_{\gamma_1 + \gamma_2 + \ldots + \gamma_n} f(z)\, dz := \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz.$$



<u>Theorem</u> Let $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$ be a piecewise C^1 curve, with range in $A \subseteq \mathbb{C}$ open. Let $f : A \to \mathbb{C}$ continuous. Then

(1)

$$\int_{-\gamma} f(z)\ \mathrm{d}z = -\int_{\gamma} f(z)\ \mathrm{d}z$$

proof: $\int_{-\gamma_j} f(z)\ \mathrm{d}z = -\int_{\gamma_j} f(z)\ \mathrm{d}z$. Now sum over j.

(2) If $\exists$ antiderivative $F : A \to \mathbb{C}$ with $F' = f$ then

$$\int_{\gamma} f(z)\ \mathrm{d}z = F(Q) - F(P)$$

where Q is the terminal point of of γ and P is the initial point.
proof:

$$\int_{\gamma} f(z)\ \mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\ \mathrm{d}z$$

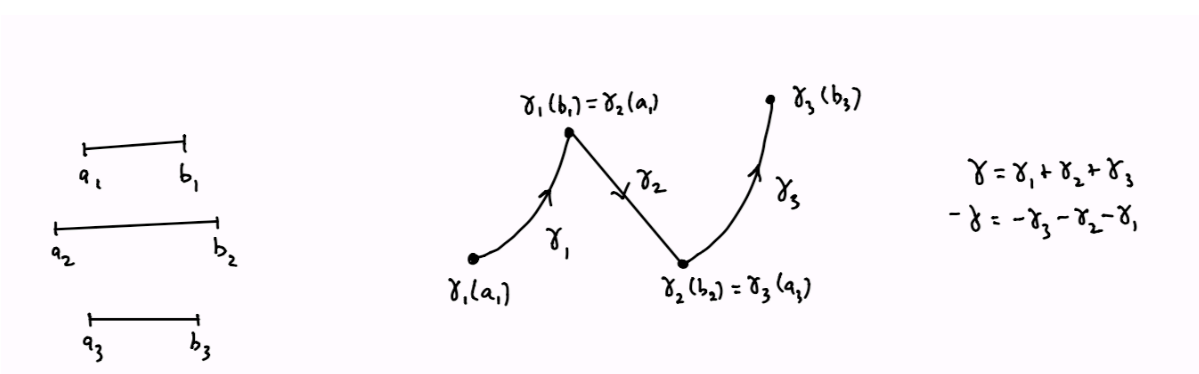
$$= \sum_{j=1}^{n} F\left(\gamma_j(b_j)\right) - F\left(\gamma_j(a_j)\right)$$

$$= -F\left(\gamma_1(a_1)\right) + F\left(\gamma_1(b_1)\right) - F\left(\gamma_2(a_2)\right) + F\left(\gamma_2(b_1)\right) - F\left(\gamma_3(a_3)\right) + F\left(\gamma_3(b_3)\right) + \ldots$$
$$- F\left(\gamma_n(a_n)\right) + F\left(\gamma_n(b_n)\right)$$

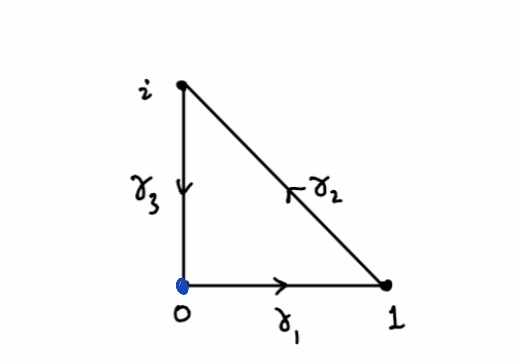
$$F(Q) - F(P)$$

(telescoping series).

(3) $\left| \int_{\gamma} f(z)\ \mathrm{d}z \right| = \left| \sum_{j=1}^{n} \int_{\gamma_j} f(z)\ \mathrm{d}z \right| \leq \sum_{j=1}^{n} \left| \int_{\gamma_j} f(z)\ \mathrm{d}z \right| \leq \sum_{j=1}^{n} \int_{\gamma_j} |f(z_j)|\,|dz_j|$.



Examples: Compute the following. Recall that particular parameterizations don't matter, just the directions of the curves.



parameterization | Green's Theorem | FTC

$\int_\gamma 1 \, dz$

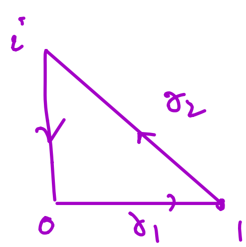
$\iint 0 \, dA$

$\int_{\gamma_1+\gamma_2+\gamma_3} 1 \, dz = z\Big]_0^0 = 0$

$\int_\gamma z \, dz$

$\iint 0 \, dA$

$\int z \, dz = \frac{z^2}{2}\Big]_0^0 = 0$



$\int_\gamma \bar{z} \, dz =$

$\int_\gamma (x-iy)(dx+idy)$

$= \int_\gamma \underbrace{x}_{P}\,dx + \underbrace{y}_{Q}\,dy + i\int_\gamma \underbrace{-y}_{P}\,dx + \underbrace{x}_{Q}\,dy$

$Q_x - P_y = 0$ $\qquad$ $Q_x - P_y = 2$

Green

$\iint 0 \, dA + i\iint 2 \, dA$

$= i \cdot 2 \cdot \frac{1}{2} = i$

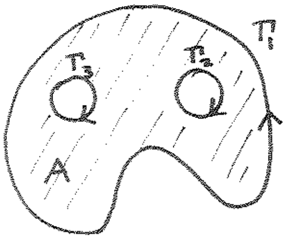
XXX

Green's Theorem

(This is just one of the vector calculus "FTC"'s, and in fact one can understand all of them as special cases of a general theorem) called Stokes' Theorem

Let $\langle P(x,y), Q(x,y)\rangle$ be a vector field, C^1 on an open domain containing the set A and its boundary ∂A

Orient $\Gamma = \partial A$ so that A is "on the left" as you traverse Γ:



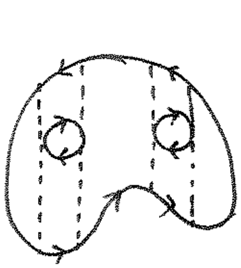
then $$\boxed{\oint_{\partial A} P\,dx + Q\,dy = \iint_A (Q_x - P_y)\, dA}$$

<u>proof</u>: ① $\displaystyle\oint_{\partial A} P\,dx = \iint_A -P_y\, dA$, ② $\displaystyle\oint_{\partial A} Q\,dy = \iint_A Q_x\, dA$

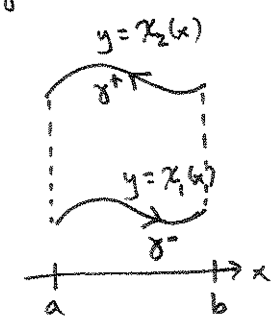
(unfortunately, I have used A for the region, and dA for dxdy — these uses of "A" are entirely independent)

① + ② = Green's Thm

① Chop up A into "vertical simple" subregions:



each region:



iterate the area integral:

$$\int_a^b \int_{\chi_1(x)}^{\chi_2(x)} -P_y(x,y)\,dy\,dx$$

$$= \int_a^b -P(x,y)\Big]_{y=\chi_1(x)}^{y=\chi_2(x)} dx$$

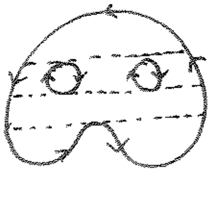
$$= \int_a^b -P(x,\chi_2(x))\,dx + P(x,\chi_1(x))\,dx$$

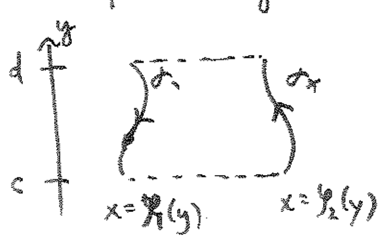
$$= \int_{\gamma^+} P\,dx + \int_{\gamma^-} P\,dx$$

add these identities for each region, and use the fact that double integrals and line integrals are additive with respect to partitioning.

Deduce $\displaystyle\iint_A -P_y\,dA = \oint_{\partial A} P\,dx$ ∎

② Chop A into "horizontal simple" subregions





$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} Q_x\,dx\,dy = \int_c^d Q(x,y)\Big]_{x=\psi_1(y)}^{\psi_2(y)} dy$$

$$= \int_c^d Q(\psi_2(y),y)\,dy - Q(\psi_1(y),y)\,dy$$