Math 4200
Wednesday September 18

Chapter 2: Complex integration.
- Leads to Cauchy Integral Formula and magic theorems which result:
  - Liouville's Theorem: Bounded entire functions are constant.
  - Fundamental Theorem of Algebra: every degree n polynomial has n (complex) roots, counting multiplicity.
  - Magic ways to compute many definite integrals (contour integration).

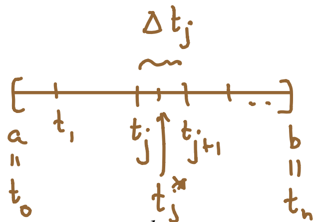
Announcements:

- midterm two weeks from today (Oct 2), probably thru §2.3
- next hw. on last page (2.1 - 2.2) (I may still add an extra credit problem)
- Monday office hrs extended 12:50-2:50 (T. still 11:50-12:40 W. still 12:50-1:50)
- 2.1-2.2. today

A1 Def: For $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{C}$ continuous, $f(t) = u(t) + i\,v(t)$, with $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t) + i\,v(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t.$$

It is useful for estimates to note that since u, v are continuous on $[a,b]$ they are uniformly continuous - and you proved in Math 3210 that in this case definite integrals are limits of Riemann sums for partionings P of $[a,b]$, as the "norm" of the partition approaches zero: For



$$P := \ a = t_0 < t_1 < \ldots < t_n = b$$
$$t_j \le t_j^* \le t_{j+t}, \quad \Delta t_j = t_{j+1} - t_j$$
$$\|P\| := \max \Delta t_j\,,$$

$$\int_a^b u(t)\,\mathrm{d}t = \lim_{\|P\| \to 0} \sum_j u(t_j^*)\Delta t_j, \qquad \int_a^b v(t)\,\mathrm{d}t = \lim_{\|P\| \to 0} \sum_j v(t_j^*)\Delta t_j$$

so also

A2 Def:

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{\|P\| \to 0} \sum_j u(t_j^*)\Delta t_j + i \lim_{\|P\| \to 0} \sum_j v(t_j^*)\Delta t_j = \lim_{\|P\| \to 0} \sum_j f(t_j^*)\Delta t_j.$$

<u>Example 1</u>: Use Calc 1 FTC to compute

$$\int_0^{\frac{\pi}{2}} -2\sin(t)\cos(t) + i\left(\cos^2(t) - \sin^2(t)\right)\,\mathrm{d}t.$$

$$= \int_0^{\pi/2} \underbrace{-2\sin t\cos t}_{-\sin 2t}\,dt + i\int_0^{\pi/2} \underbrace{\cos^2 t - \sin^2 t}_{\cos 2t}\,dt$$

$$= \left.\frac{\cos 2t}{2}\right]_0^{\pi/2} + i\left.\frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$= -\tfrac{1}{2} - \tfrac{1}{2} + i\,0$$

$$= -1$$

easy!

Use *A2* and the triangle inequality on Riemann sums to prove the important integral estimate which bounds the modulus of definite integrals in terms of the integrals of their modulus:

A3 Theorem

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\mathrm{d}t.$$

$$\| \quad \left| \lim_{\|P\|\to 0} \sum_j f(t_j^*)\Delta t_j \right|$$

$$= \lim_{\|P\|\to 0} \left| \sum_j f(t_j^*)\Delta t_j \right|$$ because $|w|$ is a cont fcn.

$$\le \lim_{\|P\|\to 0} \sum_j |f(t_j^*)|\Delta t_j$$ Δ inequality.

$$= \int_a^b |f(t)|\,dt$$ (just like 3210, after polling class)

A4 Fundamental Theorem of Calculus for $f\colon [a, b]\to\mathbb{C}$: Let $u, v : [a, b]\to\mathbb{R}$ continuous, $f(t) = u(t) + i\,v(t)$, $F(t)$ such that $\underline{F'(t) = f(t)}$. Then

$F(t) = U(t) + i V(t)$
$F'(t) = u(t) + i v(t)$

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

$$\| \quad \int_a^b u(t) + i\,v(t)\,dt$$

$$\| \quad \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$$\| \quad \| \quad U(b) - U(a) + i\,(V(b) - V(a))$$ Calc 1 FTC.

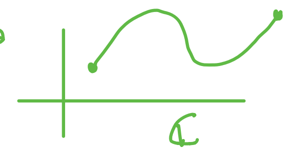
$$F(b) - F(a)$$

now it gets interesting...

B1 Def Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then the *complex line integral* or *contour integral*

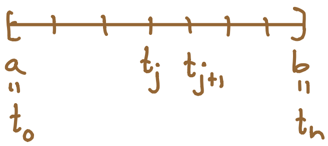
$$\int_\gamma f(z)\,\boxed{dz} := \int_a^b f(\gamma(t))\,\boxed{\gamma'(t)\,dt}$$

γ , f



where we use the definition *A1* on the previous page to compute the integral on the right. Note, we have substituted $z = \gamma(t)$ and used the differential substitution, $dz = \gamma'(t)dt$ into the integrand.

B2 In the case that $\gamma'(t) \neq 0$ for any t it follows from the continuity of $|\gamma'(t)|$ that $|\gamma'(t)| \geq \delta > 0$ on $[a, b]$. And in this case the complex line integral above can be realized as a limit which explains the geometry of what's going on:



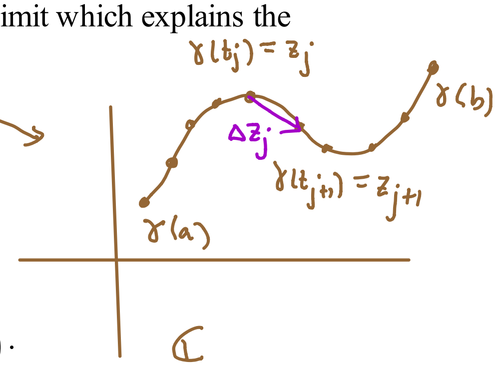
γ

$$\int_\gamma f(z)\,dz = \lim_{\max\{|\Delta z_j|\} \to 0} \sum_{j=0}^{n-1} f(z_j)\Delta z_j ,$$

$$P := \quad a = t_0 < t_1 < \ldots < t_n = b;$$

$$\Delta t_j = t_{j+1} - t_j \qquad \|P\| := \max \Delta t_j$$

$$z_j = \gamma(t_j), \quad \Delta z_j = z_{j+1} - z_j = \gamma(t_{j+1}) - \gamma(t_j).$$



The reason this is true is that by the 3220 affine approximation formula for the C^1 curve γ,

$$\gamma(t_{j+1}) - \gamma(t_j) = \gamma'(t_j)\Delta t_j + \varepsilon(t_j)\Delta t_j$$

where one can show that the $|\varepsilon(t)| \to 0$ uniformly as $\|P\| \to 0$ because γ is continuously differentiable. Also, because $M \geq |\gamma'(t)| \geq \delta$ the condition that $\max\{|\Delta z_j|\} \to 0$ in $\mathbb{C}$ is equivalent to the $\|P\| \to 0$ in $[a, b]$, also because of the approximation formula. So,

$$\lim_{\max\{|\Delta z_j|\} \to 0} \sum_{j=0}^{n-1} f(z_j)\Delta z_j$$

$$= \lim_{\|P\| \to 0} \sum_j f(\gamma(t_j))\,(\gamma(t_{j+1}) - \gamma(t_j))$$

$$= \lim_{\|P\| \to 0} \sum_j f(\gamma(t_j))\,(\gamma'(t_j)\Delta t_j + \varepsilon(t_j)\Delta t_j)$$

$$= \lim_{\|P\| \to 0} \sum_j f(\gamma(t_j))\,\gamma'(t_j)\Delta t_j \quad + \lim_{\|P\| \to 0} \left|\sum_j f(\gamma(t_j))\,\varepsilon(t_j)\,\Delta t_j\right|$$

$$= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt\,.$$

$$\leq \sum_j |\quad|$$

$$\leq \sum_j M\,\varepsilon\,\Delta t_j$$

$$= M\varepsilon(b-a) \qquad |f(\gamma(t))| \leq M$$

$\varepsilon(t_j) \leq \varepsilon \to 0$ as $\|P\| \to 0$

<u>Example 2</u>: Let $\gamma(t) = e^{it}$, $0 \le t \le \frac{\pi}{2}$, $f(z) = z$. Compute

$\gamma'(t) = ie^{it}$

$$\int_\gamma f(z)\,dz. = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$f(\gamma(t)) = e^{it}$

Sketch. Do you think you would get the same answer if you followed the same quarter circle in the same direction, but with a different parameterization? What if you reversed direction? Could you explain why?

yes see the Riemann Sum argument

$$\int_0^{\pi/2} e^{it}\, ie^{it}\,dt$$

get opposite see Riemann Sum arg. Δz_j's change sign.

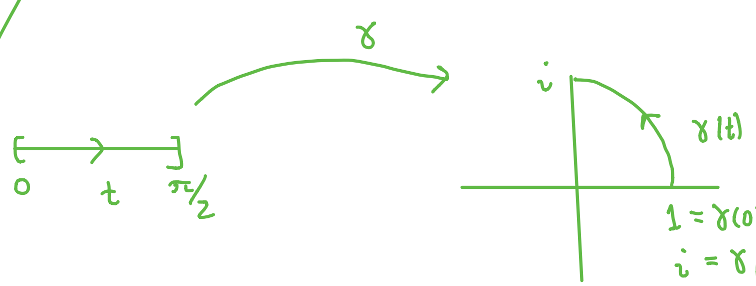
$\gamma(t) = \cos t + i\sin t$

$\gamma'(t) = -\sin t + i\cos t$

$= i(\cos t + i\sin t)$

$\frac{d}{dt} e^{at} = ae^{at}$

$t \in \mathbb{R}$



$1 = \gamma(0) = e^{i0}$

$i = \gamma(\pi/2) = e^{i\pi/2}$

$$\int_0^{\pi/2} ie^{i2t}\,dt$$

or FTC

$$\left. \frac{\cancel{i}\,e^{i2t}}{2\cancel{i}} \right]_0^{\pi/2} = \frac{1}{2}\left(e^{i\pi} - e^{0}\right)$$

$= \frac{1}{2}(-1-1)$

$= -1.$

$$\int_0^{\pi/2} \underbrace{i(\cos 2t + i\sin 2t)}\,dt$$

$-\sin 2t + i\cos 2t\,dt$

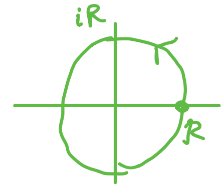
<u>Example 1</u>

ans $= -1$

$R > 0$

Example 3: For the radius R circle traversed counterclockwise, $\gamma(t) = R\,e^{i\,t}$, $0 \le t \le 2\pi$, compute

$$\int_\gamma \frac{1}{z}\,dz.$$



come back to this.

$$\int_0^{2\pi}$$

B3 Integral estimate: Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$\le \int_a^b |f(\gamma(t))\gamma'(t)|\,dt \qquad (A3)$$

$$= \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

Def Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. Then

$$\int_\gamma |f(z)|\,|dz| := \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

Using the definition, we see that the shorthand for the integral estimate in *B3* is

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|.$$

Note that $|dz| = |\gamma'(t)|\,dt$ is the element of arclength.

Example 4: In Example 2, you showed that for $\gamma(t) = e^{i\,t}$, $0 \le t \le \frac{\pi}{2}$,

$$\int_\gamma z\,dz = -1.$$

Compute

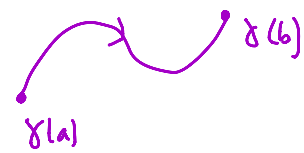
$$\int_\gamma |z|\,|dz|$$

and verify the integral estimate *B3*.

B3 <u>Theorem</u> FTC for complex line integrals Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. <u>If</u> f has an analytic antiderivative in A, i.e. $F' = f$, then complex line integrals only depend on the endpoints of the curve γ, via the formula

$$\int_\gamma f(z)\,\mathrm{d}z := F(\gamma(b)) - F(\gamma(a))$$

<u>proof</u>:



$$\| $$

$$\int_a^b \underbrace{f(\gamma(t))\,\gamma'(t)}_{F'(\gamma(t))\,\gamma'(t)}\,dt$$

part a. (A4)

chain rule for curves $\quad \int_a^b \frac{d}{dt}F(\gamma(t))\,dt \overset{!}{=} F(\gamma(t))\Big]_a^b \quad \blacksquare$

<u>Example 5</u>: Redo Example 2 using the FTC: $\gamma(t) = e^{it}$, $0 \le t \le \frac{\pi}{2}$, $f(z) = z$,

$$\int_\gamma z\,\mathrm{d}z = \frac{z^2}{2}\Big]_{\gamma(0)}^{\gamma(\pi/2)} = \frac{z^2}{2}\Big]_{1=e^{i0}}^{i=e^{i\pi/2}} = \frac{i^2}{2} - \frac{1}{2} = -\frac{1}{2} - \frac{1}{2} = -1$$

$f(z) = z$

$F(z) = \frac{z^2}{2}$

<u>Example 6:</u> Could you redo example 3 with the FTC? For the radius R circle traversed counterclockwise, $\gamma(t) = R\,e^{it}$, $0 \le t \le 2\pi$, compute

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z.$$

The connection between contour integrals and Calc 3 line integrals:

Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve. write

$$\gamma(t) = x(t) + i\,y(t),$$
$$f(z) = u(x, y) + i\,v(x, y).$$

Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

$$= \int_a^b (u(x(t), y(t)) + i\,v(x(t), y(t)))(x'(t) + i\,y'(t))\,\mathrm{d}t$$

$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)dt \; + i\int_a^b v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)dt$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy.$$

On Friday we'll combine this Calc 3 line integral way of writing contour integrals with Calc 3 Green's Theorem, for some interesting results.

Math 4200-001
Week 5 concepts and homework
2.1-2.2
Due Wednesday September 25 at start of class.

2.1 2ac, 3, 5, 10, 11, 13, 14;
2.2 1ad, 2 (prove with FTC!), 3, 4, 5, 6, 8, 9, 10.

Hint: In many of these problems the fundamental theorem of Calculus for contour integrals lets you find the answer very quickly if you can find an antiderivative on an appropriate (and possibly but not necessarily branched,) domain.

I may add an extra credit problem on or before Friday.