Math 4200
Monday September 16

1.6 differentiation and mapping of elementary functions and branches of their inverses, and compositions of all of these.

Announcements:

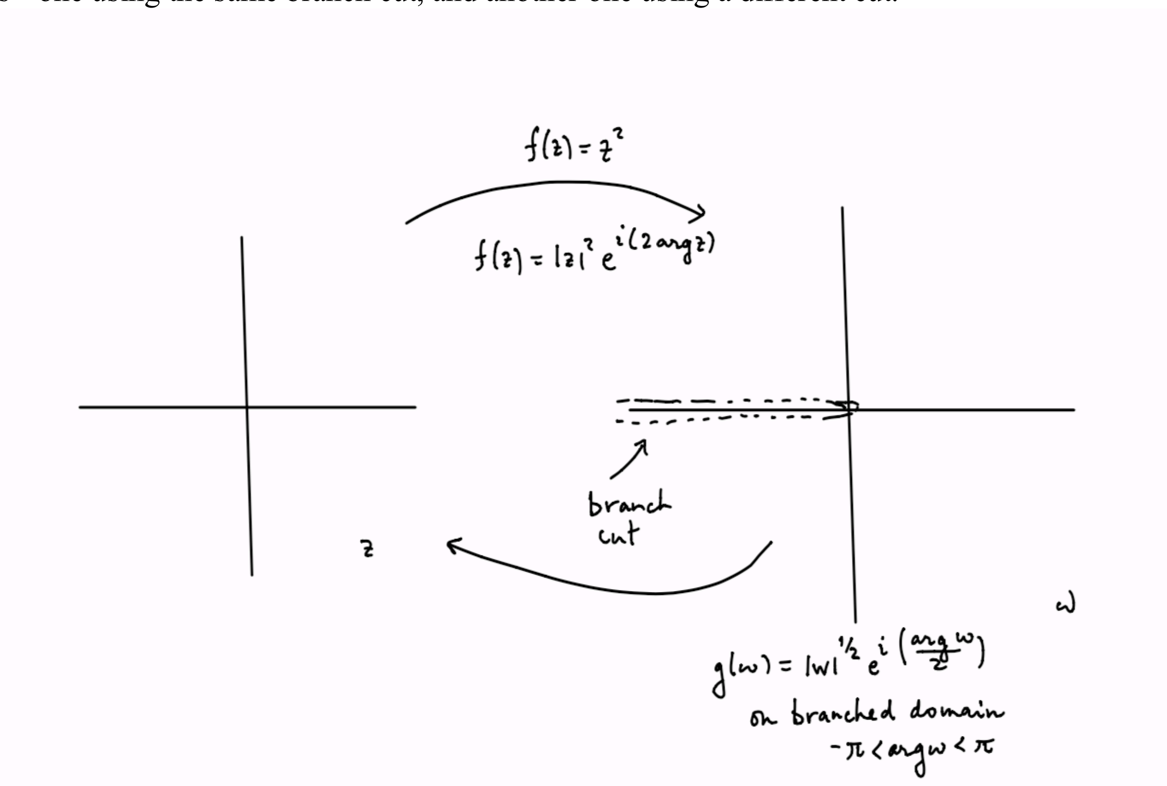
On Friday we talked about the fact that entire functions (analytic functions defined on all of $\mathbb{C}$) map onto almost all of $\mathbb{C}$, so that " branches" of (partial) inverse functions can be defined on connected open domains which are the complement in $\mathbb{C}$ of *branch points* (which are either points not in the range of f or images of points where $f' = 0$), and *branch cuts* connecting these branch points (which may also include the "point" at infinity).

At the end of class we partially completed the first example below. Let's pick up there.

Example 1) $f(z) = z^2$, $g(w) = \sqrt{w}$ (for some branch choice). Note for any branch choice of g,

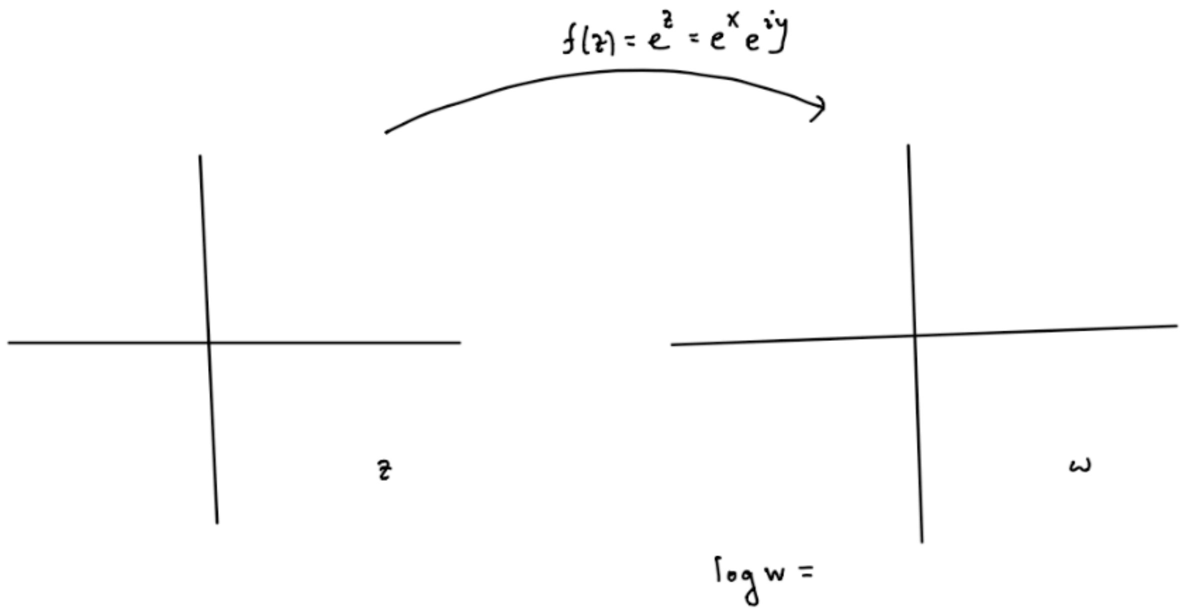
$$f(g(w)) = w$$
$$f'(g(w))g'(w) = 1$$
$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{2\,g(w)} = \frac{1}{2}w^{-\frac{1}{2}}.$$

Describe the range of the branch of the square root function defined below. Write down two other branch choices - one using the same branch cut, and another one using a different cut.



Example 2) $f(z) = e^z = e^x e^{iy}$. Find and illustrate the branch of $g(w) = \log(w)$ where we choose $-\pi < arg(w) < \pi$. Note that for any branch choice,

$$f(g(w)) = w$$
$$f'(g(w))g'(w) = 1$$
$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{e^{g(w)}} = \frac{1}{e^{\log(w)}} = \frac{1}{w}.$$



If we compose entire functions with themselves and inverse branches of others, the zoo of functions and branch domain choices expands. We've already had the discussion of how to define some of these functions, and now we know how to use the chain rule to compute their derivatives. You do computations and branched-domain finding in section 1.6 homework problems related to interesting composition functions. One of the most challenging in terms of finding the appropriate branched domain is for the inverse sine function, $\arcsin(z)$ that agrees with the usual Calculus inverse sine function whose domain of differentiability is the interval $(-1, 1)$, with range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. You should be able to work out the formula for $\arcsin(z)$ using the logarithm function and the quadratic formula, and based on similar earlier homework problem ideas, but specifying the branch choice and domain carefully is some serious work.

3) $f(z) = z^a$, $a \in \mathbb{C}$ is defined by

$$f(z) = z^a = e^{a \log z}$$

is defined up to a choice of branch domain for the logarithm function as on the previous page.

And we get the derivative formula we expect

$$\frac{d}{dz} z^a = \frac{d}{dz} a\, z^{a-1}$$

defined on the same branched domain (which we should verify).

As we checked before, this general definition of z^a is consistent: when $a = n \in \mathbb{Z}$ all choices of $e^{n \log z}$ lead to the usual z^n power function. And if $a = \frac{1}{n}$ then all branch choices of $e^{\frac{1}{n}\log(z)}$ lead to one of the n^{th} roots of z, of which there are n.

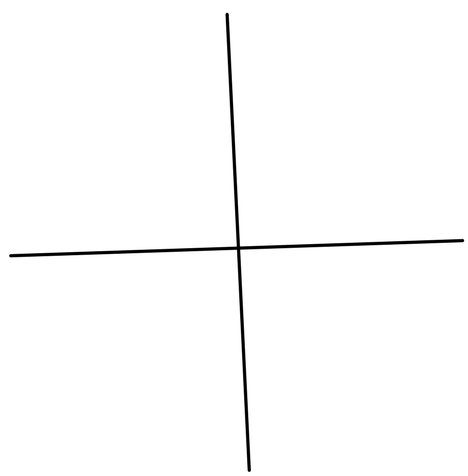
4) <u>Example</u> Find a definition and branched domain for

$$f(z) = \sqrt{z^2 - 1}\,.$$

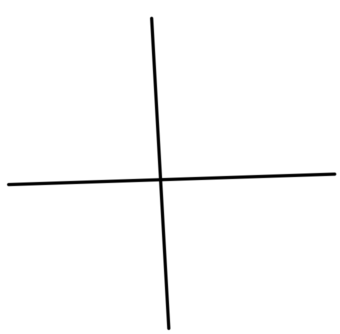
(In your homework you will do an analogous procedure for $g(z) = \sqrt{z^3 - 1}$.) Begin by identifying branch points based on where f or f' cannot be not defined.

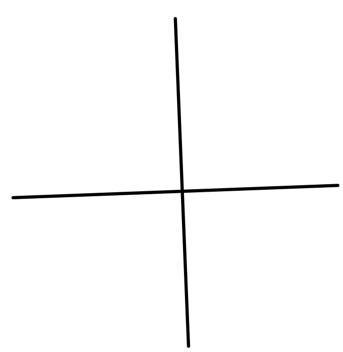
Then

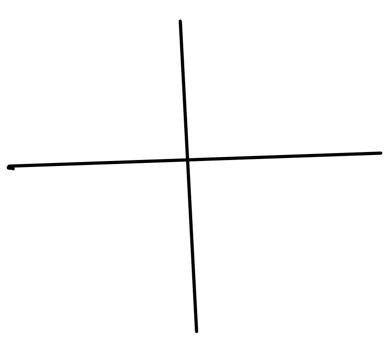
a) Writing $f(z) = \sqrt{z^2 - 1} = \sqrt{z - 1}\sqrt{z + 1}$ leads to one possible way of proceding.

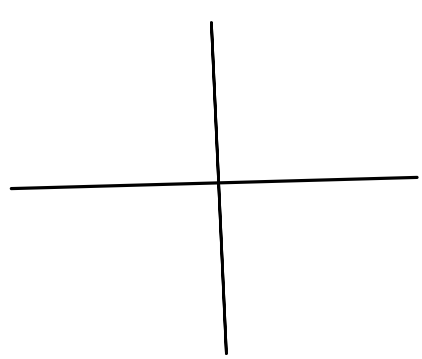


b) Considering f as a composition, $f(z) = g \circ h(z)$ with $h(z) = z^2 - 1$ and $g(w) = \sqrt{w}$ recovers the first branched domain, but leads to a choice with only a finite branch cut, as well as the original one.



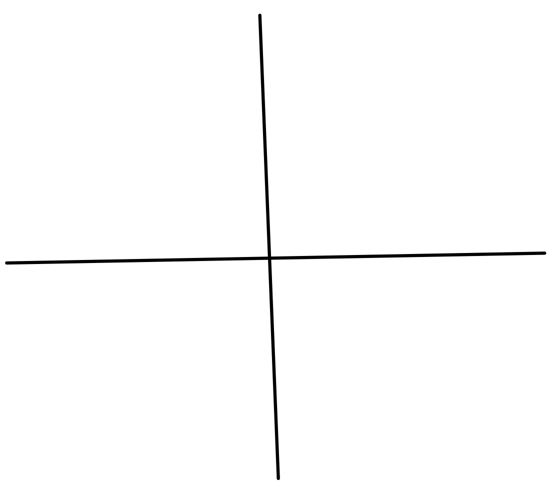


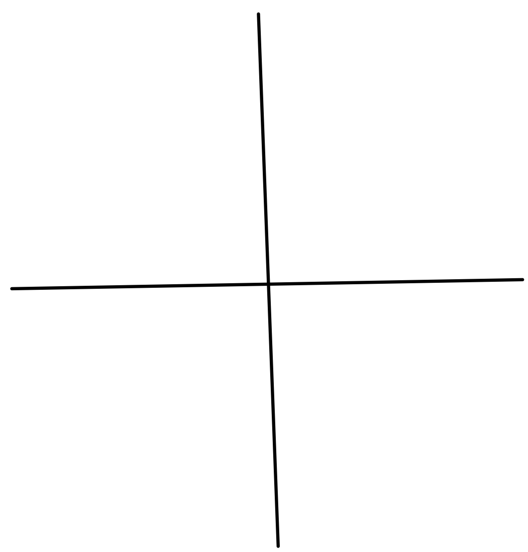




Example Find a definition and branched domain for

$$f(z) = \sqrt{e^z + 1}\,.$$





Appendix 1 from Friday discussion: partial proof of key point for harmonic conjugate construction.

Theorem Let A be an open simply connected domain in $\mathbb{R}^2$. Let $[P, Q]$ be a C^1 vector field defined on A. Then there is a function $v \in C^2(A)$ so that

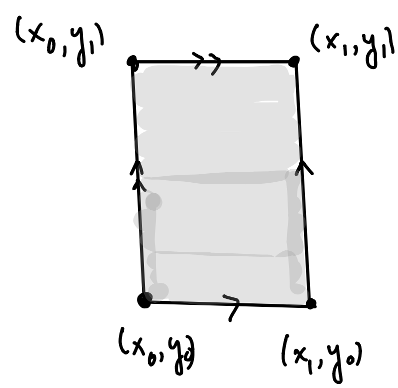
$$v_x = P(x, y), \quad v_y = Q(x, y)$$

if and only if the curl of the vector field is zero:

$$P_y = Q_x.$$

This condition is necessary since if v exists then $v_{xy} = P_y$ and $v_{yx} = Q_x$.

Local proof added: (Once we've carefully defined simply-connected domains in Chapter 2, the global theorem in a simply connected region from this local version.) Let P, Q be real differentiable, with continuous partials in $B_r((x_0, y_0))$, $r > 0$, and satisfying the "zero curl" condition $P_y = Q_x$. Let $v(x_0, y_0)$ be any chosen constant. Then $\forall$ points $(x_1, y_1) \in B_r((x_0, y_0))$ define $v(x, y)$ in a way which would be consistent with $P = v_x$, $Q = v_y$ if we already knew the function $v(x, y)$. There are two ways to do this using the fundamental theorem of Calculus, and following sides of a rectangle. The curl condition ensures that both routes yield the same value:



$$(1) \qquad v(x_1, y_1) = v(x_0, y_0) + \int_{x_0}^{x_1} P(x, y_0)\, dx + \int_{y_0}^{y_1} Q(x_1, y)\, dy$$

$$(2) \qquad v(x_1, y_1) = v(x_0, y_0) + \int_{y_0}^{y_1} Q(x_0, y)\, dy + \int_{x_0}^{x_1} P(x, y_1)\, dx$$

The two formulas agree iff the difference of their right hand sides equals zero:

$$\int_{x_0}^{x_1} P(x, y_0) - P(x, y_1)\, dx + \int_{y_0}^{y_1} Q(x_1, y) - Q(x_0, y)\, dy = 0$$

iff

$$\int_{x_0}^{x_1} \left(-\int_{y_0}^{y_1} P_y(x, y)\, dy \right) dx + \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} Q_x(x, y)\, dx \right) dy = 0.$$

This last equality holds because $-P_y + Q_x = 0$ in the rectangle.

Finally, using (1) to compute v_{y_1} we see $v_{y_1}(x_1, y_1) = Q(x_1, y_1)$; and using (2) we compute $u_{x_1}(x_1, y_1) = P(x_1, y_1)$ QED.

Appendix 2, from last Wednesday's notes:
<u>Theorem</u> Let A be an open connected set in $\mathbb{C}$, $f: A - \mathbb{C}$ analytic, with $f'(z) = 0 \ \forall \ z \in A$. Then f is constant.

<u>proof:</u> Since $f'(z) = f_x = -i f_y$ the hypothesis is equivalent to saying that the partial derivatives of $u(x, y) = \mathrm{Re}(f(x + i y), v(x, y) = \mathrm{Im}(f(x + i y)$ are identically zero in A. Pick $a \in A$. Consider the set

$$U := \{z \in A \mid f(z) = f(a)\}.$$

U is closed in A because $\{f(a)\}$ is a closed set, f is continuous, and $U = f^{-1}(\{f(a)\})$.

But the idea of the argument on the previous page also shows that U is open: Let $z_0 \in U, \mathrm{D}(z_0 ; r) \subseteq A$. Then for $z_1 \in \mathrm{D}(z_0 ; r)$ we compute

$$u(x_1, y_1) = u(x_0, y_0) + \int_{x_0}^{x_1} u_x(x, y_0)\, \mathrm{d}x + \int_{y_0}^{y_1} u_y(x_1, y)\, \mathrm{d}y = u(x_0, y_0) = \mathrm{Re} f(a)$$

$$v(x_1, y_1) = v(x_0, y_0) + \int_{x_0}^{x_1} v_x(x, y_0)\, \mathrm{d}x + \int_{y_0}^{y_1} v_y(x_1, y)\, \mathrm{d}y = v(x_0, y_0) = \mathrm{Im} f(a).$$

Thus $\mathrm{D}(z_0 ; r) \subseteq U$, so U is open. Since U is open and closed in A, its complement $V := A \setminus U$ is as well. Since U is nonempty and A is connected, V must be empty, i.e. $f(z) = f(a) \ \forall \ z \in A$.

QED