Math 4200
Friday September 13

1.5: harmonic functions, harmonic conjugates; 1.6 differentiation and mapping of elementary functions and branches of their inverses.

Announcements: we'll complete today's notes first, and then take care of Wed. loose end:

If $f'(z) \equiv 0$ in open & connected domain, then $f(z) \equiv$ constant.

(you can use this fact in hw 1.5.16)

Harmonic functions and harmonic conjugates.

Let $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$ be analytic in an open domain A, and assume u, v have continuous first and second partial derivatives. (The shorthand for this is $u, v \in C^2(A)$.) Then from Cauchy Riemann

continuous thru 2nd order partials

$$\boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}}$$

we compute

the Laplacian of u

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

(Recall from 3220 or multivariable calculus that $v_{yx} = v_{xy}$ when all second partial derivatives are continuous.)

Def Let $U(x, y)$ be a C^2 function in a domain $A \subseteq \mathbb{R}^2$. Then U is *harmonic* in A if it satisfies the partial differential equation

$$U_{xx} + U_{yy} = 0.$$

Def The partial differential equation above is called *Laplace's equation*.

(Harmonic functions are important in pure and applied math, as well as in physics. Also harmonic functions of three or more variables. If you've taken any class on partial differential equations or electro-magnetism, you've seen harmonic functions before.)

Exercise: Let $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$ be analytic in an open domain A, with $u, v \in C^2(A)$. Is $v(x, y)$ harmonic as well?

$$v_{xx} + v_{yy}$$
$$= -u_{yx} + (u_x)_y = 0$$

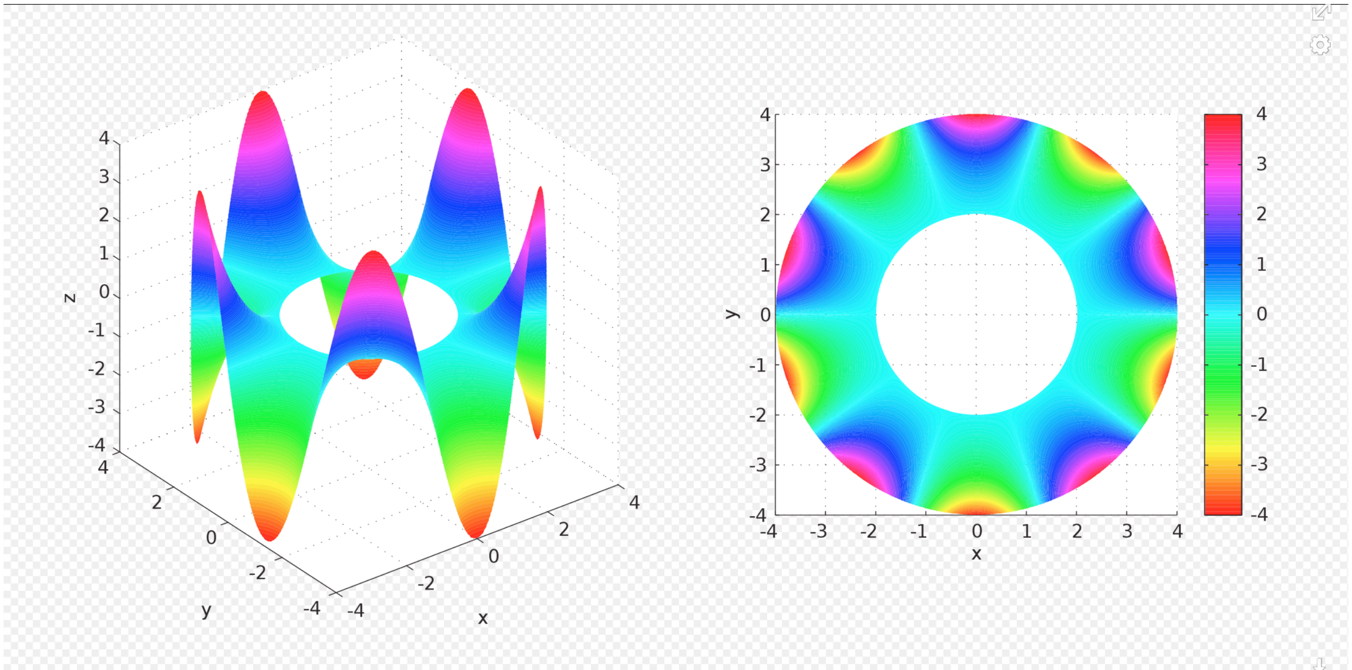
$$v_x = -u_y$$
$$v_y = u_x$$

yes.

OR $-i f(z)$ is analytic, its real part is $v(x,y)$, use previous page

$= -i\,u(x,y) + v(x,y)$

from the Wikipedia page on harmonic functions...



A harmonic function defined on an annulus.

More details

Def Let $A \subseteq \mathbb{C}$ open, and let $u \in C^2(A)$ be a harmonic function. A function $v(x, y)$ such that

$$f(z) = f(x + i y) = u(x, y) + i v(x, y)$$

is analytic in A is called a *harmonic conjugate* to $u(x, y)$.

Theorem If $u(x, y) \in C^2(A)$ where A is an open *simply connected* domain. (A domain is called simply connected if its connected and "has no holes". We'll discuss this concept more carefully in the next chapter.) Then there exists a harmonic conjugate $v(x, y)$ to $u(x, y)$, unique up to an additive constant.

proof: $u \in C^2(A)$, $u_{xx} + u_{yy} = 0$ is given. The system for finding $v(x, y)$ has to be consistent with the Cauchy-Riemann equations for f:

$$v_x = P(x, y) \qquad (= -u_y)$$
$$v_y = Q(x, y) \qquad (= u_x)$$

reviews these concepts
1.5.28 → last test problem
?

When you studied *conservative vector fields* and *Green's Theorem* in multivariable calculus you learned that a vector field $[P, Q]^T$ is actually the gradient of a function $v(x, y)$ locally if and only if the necessary condition that v_{xy} would equal v_{yx} holds:

$$P_y = Q_x$$

In our case, since P, Q are partials of $u(x, y)$ this integrability condition reads as

$$-u_{yy} = u_{xx}$$

which holds since u is harmonic!

Example Let $u(x, y) = x y$. Show u is harmonic. Then find its harmonic conjugate $v(x, y)$ and identify the analytic function $f(z) = u(x, y) + i v(x, y)$.

① $u_x = y \qquad u_y = x$
$u_{xx} = 0 \qquad u_{yy} = 0$
$u_{xx} + u_{yy} = 0$ ✓
u is harmonic

② find its conjugate v. ($u + iv$ is spposed to be analytic)

CR $\begin{cases} v_x = -u_y = -x \\ v_y = u_x = y. \end{cases}$ antidiff to find v

$v_x = -x$

1210 FTC $\int v_x \, dx = v(x,y) + \cancel{C}\, g(y)$

$v = \int v_x \, dx = \int -x \, dx = -\frac{x^2}{2} + \cancel{C}\, g(y)$

the "const" depends on which x-coord curve I was integrating along.

So $v(x,y) = -\frac{x^2}{2} + g(y)$

$\Rightarrow v_y = 0 + g'(y) = y \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C$ $\Big\}$ $v(x,y) = -\frac{x^2}{2} + \frac{y^2}{2} + C$

should be that
$f(z) = u + iv$ is analytic
$f(z) = xy + i\left(-\frac{x^2}{2} + \frac{y^2}{2}\right) + Ci$

$g(z) = z^2$
$= (x + iy)^2$
$= x^2 - y^2 + 2ixy$

$\Rightarrow f(z) = -\frac{i}{2} g(z)$?? $+ Ci$
$= -\frac{i}{2}\left[(x^2 - y^2) + 2ixy\right]$ ✓

Example $u(x, y) = \ln\sqrt{x^2 + y^2}$ is harmonic in $\mathbb{C} \setminus \{0\}$ (we could check directly but already know that $u(x, y) = \mathrm{Re}(\log(z))$. Use the Cauchy-Riemann equations in polar coordinates to (re)find its harmonic conjugate. This illustrates why domains which are not simply connected may not have globally-defined harmonic conjugates. ($\mathbb{C} \setminus \{0\}$ has a pin-prick hole at the origin.) CR:

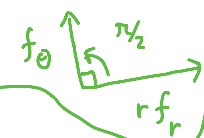
$u(r, \theta) = \ln r$

$$f_\theta = i\, r\, f_r$$
$$u_\theta = -\, r\, v_r$$
$$v_\theta = r\, u_r$$

$\frac{\partial}{\partial \theta} f(re^{i\theta}) = f'(re^{i\theta})\, r i e^{i\theta}$

$\frac{\partial}{\partial r}\left(f(re^{i\theta})\right) = f'(re^{i\theta})\, e^{i\theta}$

$ri(u_r + iv_r) = u_\theta + iv_\theta$ $\quad ri f_r = f_\theta$



$v_r = -\frac{1}{r} u_\theta = 0$

$v_\theta = r u_r = r\frac{1}{r} = 1$

θ has to be continuous

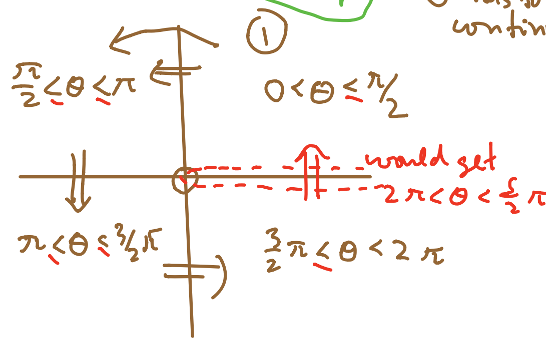
$v_\theta = 1$

$v = \int v_\theta \, d\theta = \int 1 \, d\theta = \theta$ $\cancel{(+C)}$ $+ g(r)$

$\Rightarrow v_r = 0 = g'(r) \Rightarrow g(r) = C.$

CR

$v(r, \theta) = \theta + C.$ set $= 0$



on $\mathbb{C} \setminus$ pos real axis, θ is a wonderful fcn.

1.6 The zoo of basic analytic functions, their derivatives, and branches for their inverses.

Def If $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$, then f is called *entire.*

Examples:

$f(z) = z^n,\ n \in \mathbb{Z} \setminus \{0\}$ $\qquad$ $f'(z) = n\, z^{n-1}$

$f(z) = e^z$ $\qquad$ $f'(z) = e^z$

$f(z) = \cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $\qquad$ $f'(z) = \frac{1}{2}\left(e^{iz} i + e^{-iz}(-i)\right) = \frac{i}{2}\left(e^{iz} - e^{-iz}\right)$

$$-\frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

$$= -\sin z$$

$f(z) = \sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ $\qquad$ $f'(z) = \cos z$

And one can add, multiply, and compose entire functions to create quite a zoo. The inverse function possibilities are also interesting:

Branches of analytic functions overview: If f is entire then it turns out (Picard's Theorem) that if f is not a constant function, then the range of f cannot omit more than two points in $\mathbb{C}$. Furthermore, it turns out that the zeroes of $f'(z)$ are isolated (i.e. if $f'(z_0) = 0$ then there exists $r > 0$ such that $f'(z) \neq 0\ \forall\, w \in D(z_0, r) \setminus \{z_0\}$.) So f has a local inverse function except at possibly a countable set of $z \in \mathbb{C}$.

In most cases this means one can piece these local inverses together to construct a partial "inverse" function g on a very large subdomain of $\mathbb{C}$. It will satisfy half of the inverse function condition, namely

$$f(g(z)) = z$$

And the domain of g can usually be chosen to be a connected open domain $A \subseteq \mathbb{C}$ with a just finite number of curves removed from $\mathbb{C}$ to get A. These omitted curves are called *branch cuts*, and the choice of (partial) inverse function is called a *branch* of the inverse function. Branch cuts always terminate either at ∞ (which means $|z| \to \infty$), or at finite points, and these are called *branch points*.

Examples (revisited). We'll explore more complicated examples on Monday, but these two are a good way to get acquainted with the terminology.

Example 1) $f(\boldsymbol{z}) = \boldsymbol{z}^2$, $g(w) = \sqrt{w}$ (for some branch choice). Note for any branch choice of g,

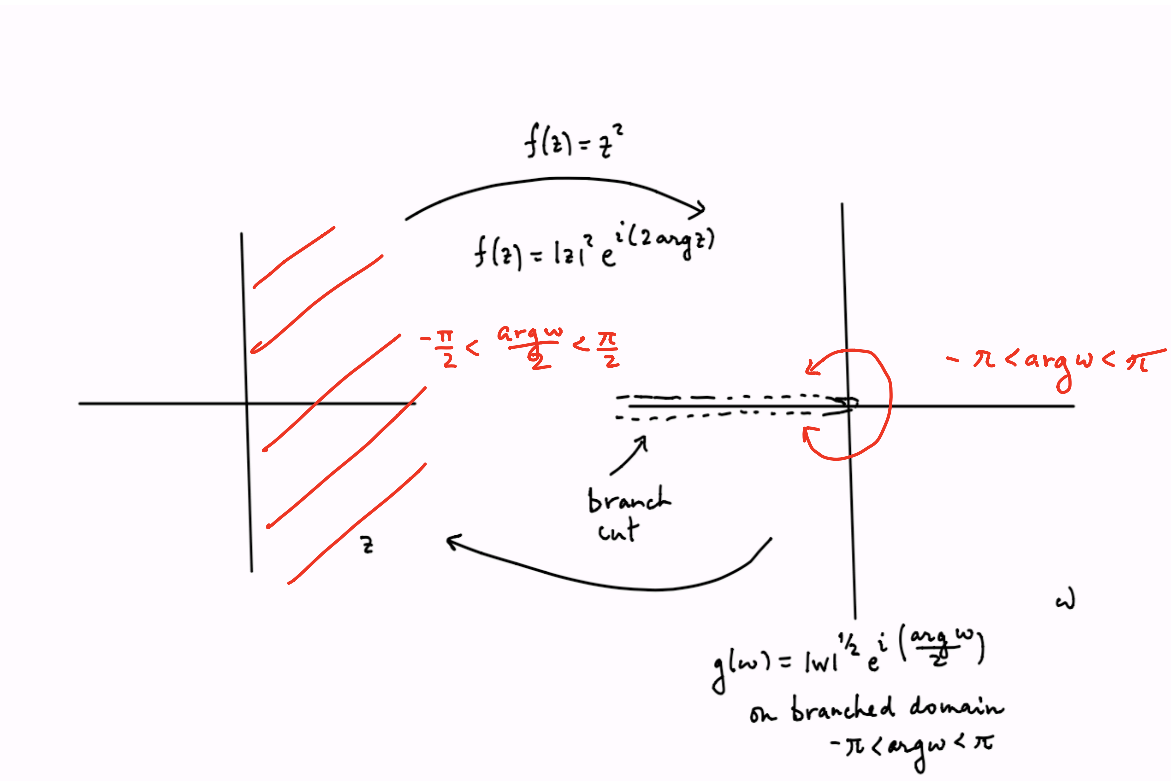
$$f(re^{i\theta}) = r^2 e^{i2\theta} = w = \rho e^{i\phi}$$

$$\rho = r^2 \qquad \phi = 2\theta + 2\pi k$$

$$r = \sqrt{\rho} \qquad \theta = \phi/2 + \pi k$$

$$f(g(w)) = w$$
$$f'(g(w))g'(w) = 1$$
$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{2\,g(w)} = \frac{1}{2}w^{-\frac{1}{2}}.$$

Describe the range of the branch of the square root function defined below. Write down two other branch choices - one using the same branch cut, and another one using a different cut.



Example 2) $f(z) = e^z = e^x e^{iy}$. Find and illustrate the branch of $g(w) = \log(w)$ where we choose $-\pi < arg(w) < \pi$. Note that for any branch choice,

$$f(g(w)) = w$$
$$f'(g(w))g'(w) = 1$$
$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{e^{g(w)}} = \frac{1}{e^{\log(w)}} = \frac{1}{w}.$$