Wednesday September 11

1.5: precise discussion of the differential map in our context, which is also used more generally for differentiable transformations between Euclidean spaces and differentiable manifolds; inverse function theorem; loose end.

Announcements

Conformal transformations and differentials discussion:

(i) The precise definition of the *tangent space* at $z_0 \in \mathbb{C}$ is the set of all *tangent vectors* there, i.e. tangent vectors to curves passing through z_0:

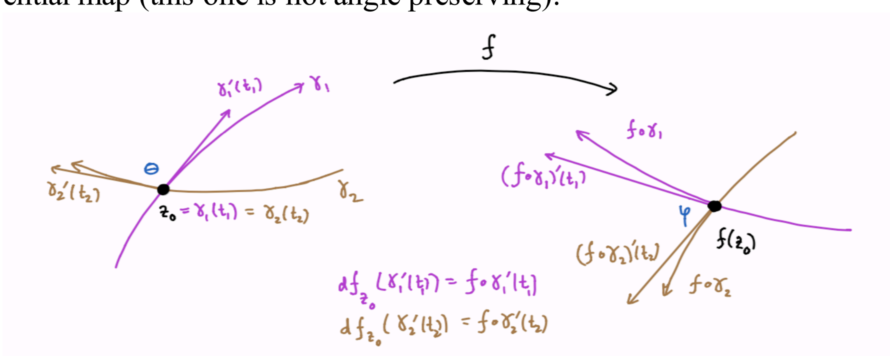
$$T_{z_0}\mathbb{C} := \left\{ \gamma'(t_0) \;\middle|\; \gamma \text{ is differentiable at } t_0 \text{ and } \gamma(t_0) = z_0 \right\}$$

(ii) If $f(z)$ is a function from $\mathbb{C}$ to $\mathbb{C}$ that arises from a real-differentiable function $F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, then the *differential of f at z_0* is defined as the transformation

$$df_{z_0}(\gamma'(t_0)) := (f \circ \gamma)'(t_0).$$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}.$$

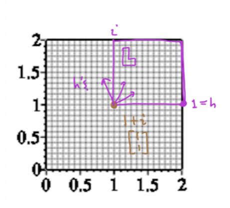
Picture of a differential map (this one is not angle preserving):

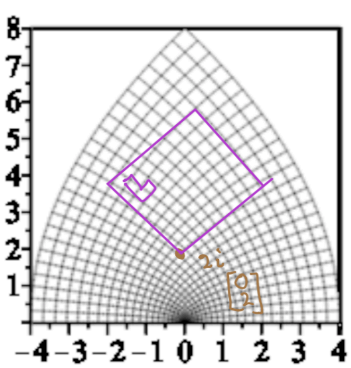


(iii) By the chain rule for curves, if $f(z)$ is complex differentiable at z_0, then

$$df_{z_0}(\gamma'(t_0)) := (f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0).$$

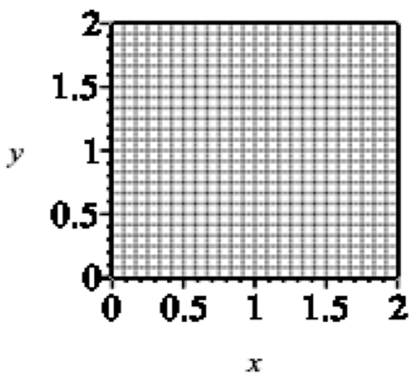
Note that multiplication by $f'(z_0)$ is the same the rotation-dilation that we discused in the context of the affine approximation formula in the first half of Monday's lecture. So the picture we drew for $f(z) = z^2$ at $z_0 = 1 + i$, and $f(z_0) = 2i$, $f'(z_0) = 2(1+i) = 2\sqrt{2}\,e^{i\frac{\pi}{4}}$, to represent the linear part of the affine transformation formula - which was only relatively accurate for small $|h|$ - is actually the precise picture for the differential map if we redefine the h's to represent all possible tangent vectors at $1 + i$

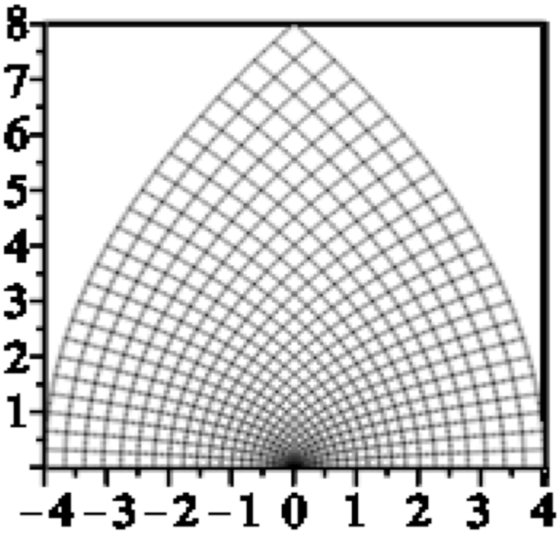




(iv) A function $f:\mathbb{C}\rightarrow\mathbb{C}$ is called *conformal* at z_0 iff its differential transformation preserves angles between tangent vectors. Since rotation-dilation transformations have this property, a function f which is complex differentiable at z_0, and for which $f'(z_0) \neq 0$, is conformal at z_0 .

Example: For $f(z) = z^2$, the coordinate curves in the domain are perpendicular everywhere they intersect. Since $f(z) = z^2$ is conformal everywhere except at $z = 0$, the images of the coordinate curves are also perpendicular everywhere except at $f(0) = 0$, where the differential map is zero.





Remark: You might wonder whether analytic functions are the only way conformal maps from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ can arise. That's almost true:

Let $F : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be real differentiable, $(x_0, y_0) \in A$. Then the differential map from the domain tangent space $T_{(x_0, y_0)}\mathbb{R}^2$ to the image tangent space $T_{F(x_0, y_0)}$ is defined analogously to what we did in the complex case, and this definition is also used for functions $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$dF_{(x_0, y_0)}(\gamma'(t_0)) := (F \circ \gamma)'(t_0).$$

From the multivariable chain rule in 3220, you know the right side is computed using the real-variables chain rule:

$$(F \circ \gamma)'(t_0) = [DF(x_0, y_0)][\gamma'(t_0)].$$

In fact, your 3220 textbook by Taylor calls the derivative matrix the "differential matrix" and writes it as $dF(x_0, y_0)$. Precisely, that so-called differential matrix is the matrix for the differential transformation of tangent spaces, as described above. (People often go back and forth between linear transformations and the matrices that describe them.)

If we write

$$F(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$$

one can show that the differential $dF_{(x_0, y_0)}$ preserves *signed* angles between tangent vectors (i.e. where counterclockwise angles up to π are positive, and clockwise ones of the same absolute values are negative), if and only if the derivative matrix $DF(x_0, y_0)$ is a rotation-dilation, i.e. if and only if the corresponding $f(z)$ *is* analytic at z_0. So talking about these sorts of conformal maps is exactly the same as talking about analytic functions with non-zero derivatives. (And the only other conformal functions arise if the differential transformation preserves *unoriented* angles but reverses *signed* angles, i.e. reverses orientation. In that case, the conformal transformation can be written as the composition of an analytic function with a reflection, namely as $f(z) = g(\bar{z})$ for an analytic g.)

Remark Gauss was interested in conformal transformations of the sphere (representing the earth) and the plane, i.e. maps of the earth. His early-on day job was as a surveyor. For example, stererographic projection maps, e.g. from the north or south pole, are conformal. Unfortunately, even though it keeps lines of longitude and latitude perpendicular, the standard rectangular maps of the earth are not conformal, and so they cause angle distortions, especially when you go towards the poles. Even better than a conformal map would be one that preserved dot products, i.e. lengths and angles. (Or scaled lengths at all points by the same constant factor. and preserved angles, so that your map could fit on a table.) Gauss proved that there is no such $\mathbb{R}^2$ map for any neighborhood on the sphere. In fact, this circle of ideas led him to develop his ideas in geometry - the branch that is now known as "Riemannian geometry", ironically, since Gauss predated Riemann.) Gauss considered this one of his most important results. Conformal maps are also important in a lot of other applications.

Another important application of the correspondence between $f: A \subseteq \mathbb{C} \to \mathbb{C}$, and $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$:

<u>Theorem</u> (Inverse function theorem) Let f be complex differentiable in a neighborhood of z_0, with $f'(z_0) \neq 0$ and $f'(z)$ continuous. Then there exist open sets U, V with $z_0 \in U$, $f(z_0) \in V$ such that $f: U \to V$ is a bijection and $f^{-1}: V \to U$ is also analytic. Furthermore

$$(f^{-1})'(f(z)) = \frac{1}{f(z)}$$

$\forall\, z \in U$.

$$F(x, y) = (u(x, y), v(x, y))$$
$$F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ A \text{ open}$$

proof: 3220 and correspondence between f, F!

$\mathbb{C}$	$\mathbb{R}^2$

$f: A \subset \mathbb{C} \to \mathbb{C}$

f analytic in a nbhd A of z_0, $f'(z)$ continuous on A, $f'(z_0) \neq 0$,

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$
$$f'(z_0) = a + bi = f_x(z_0) = -i f_y(z_0)$$
$$\left(a = u_x(x_0,y_0) = v_y(x_0,y_0) \right.$$
$$\left. b = v_x(x_0,y_0) = -u_y(x_0,y_0) \right)$$

① $\Longrightarrow$

$F: A \subset \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ real differentiable in a nbhd of (x_0,y_0)

$$DF(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$DF(x_0,y_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible ($\det = a^2 + b^2 = |f'(z_0)|^2$)

② $\Downarrow$ 3220 inverse fcn thm

$\exists\, \mathcal{U}, V$ open, $(x_0,y_0) \in \mathcal{U} \subset A$; $F(x_0,y_0) \in V$ s.t. $F: \mathcal{U} \to V$ is a bijection, and $F^{-1}: V \to \mathcal{U}$ is real differentiable.

Furthermore, by the 3220 chain rule and since

$$F^{-1}(F(x,y)) = (x,y) \quad \forall (x,y) \in \mathcal{U}$$

the matrix product of derivative matrices

$$DF^{-1}(F(x,y))\, DF(x,y) = I \leftarrow \text{identity matrix}$$

In particular,

$$DF^{-1}(F(x,y)) = [DF(x,y)]^{-1} \quad \forall (x,y) \in \mathcal{U}.$$

Note, $DF(x,y) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, so

$$DF^{-1}(F(x,y)) = \frac{1}{\alpha^2+\beta^2} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

is also a rotation-dilation

③ $\Longleftarrow$

$\exists\, \mathcal{U}, V$ open, $z_0 \in A \subset \mathcal{U}$, $f(z_0) \in V$ s.t. $f: \mathcal{U} \to V$ is a bijection, and $f^{-1}: V \to \mathcal{U}$ has real differentiable counterpart F^{-1} whose partials satisfy CR eqtns

$\Rightarrow f^{-1}$ is analytic on V.

And since $f^{-1}(f(z)) = z$

$$(f^{-1})'(f(z)) \cdot f'(z) = 1$$
$$(f^{-1})'(f(z)) = \frac{1}{f(z)}, \quad \forall z \in \mathcal{U}.$$

∎

Example Let $f(z) = \log z = \ln|z| + i\,arg(z)$. Prove $f(z)$ is analytic with $f'(z) = \dfrac{1}{z}$, away from $z = 0$ (for any continuous branch choice i.e. by specifying $arg(z)$ continuously in a neighborhood of z). Do this three ways!

1) Inverse function and chain rule.

2) Rectangular Cauchy-Riemann equations plus continuous partials. ($f_y = i f_x$ is a compact way to express CR, recoverable from the chain rule for curves, and which reminds us that df is a rotation-dilation), plus C^1.

3) Polar coordinate CR equations, plus C^1. (You worked out the CR equations in polar coordinates in your homework probably using 3220 chain rule; we can recover them quickly from the chain rule for curves, writing $f(z) = f\left(r\,e^{i\theta}\right)$.

Loose end:

<u>Theorem</u> Let A be an open connected set in $\mathbb{C}$, $f: A \to \mathbb{C}$ analytic, with $f'(z) = 0 \ \forall z \in A$. Then f is constant.

proof: