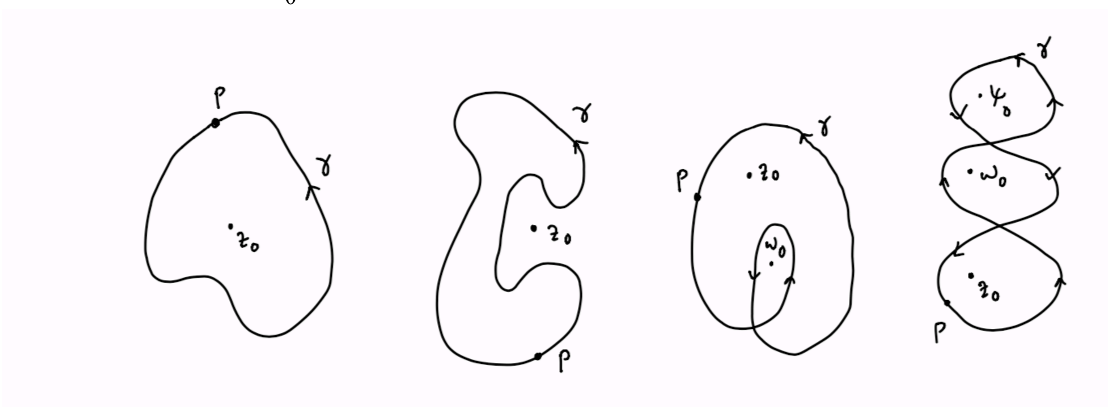
Math 4200
Friday October 4
2.4 Winding number (aka index), and Cauchy's integral formula

Announcements:

2.4 This section is about the magic fact that if a piecewise C^1 closed contour γ is given; and if $f(z)$ is analytic in an open simply-connected domain A containing γ, and if z_0 is "inside" γ, then $f(z_0)$ can be computed with an appropriate contour integral around γ. This is the *Cauchy Integral Formula* and is the basis for many amazing facts about analytic functions, and corollaries important in diverse pure and applied mathematics applications.

Step 1 What does it mean for z_0 to be "inside" γ ?



Def If γ is a continuous closed path in $\mathbb{C}$, $\gamma : [a, b] \to \mathbb{C}$, $\gamma(a) = \gamma(b) = P$, and if $z_0 \notin \gamma([a, b])$ then the *winding number* of γ about z_0, also called the *index* of γ *relative to* z_0, is how many times γ winds around z_0 in the counterclockwise direction. We write $I(\gamma ; z_0)$ for this integer.

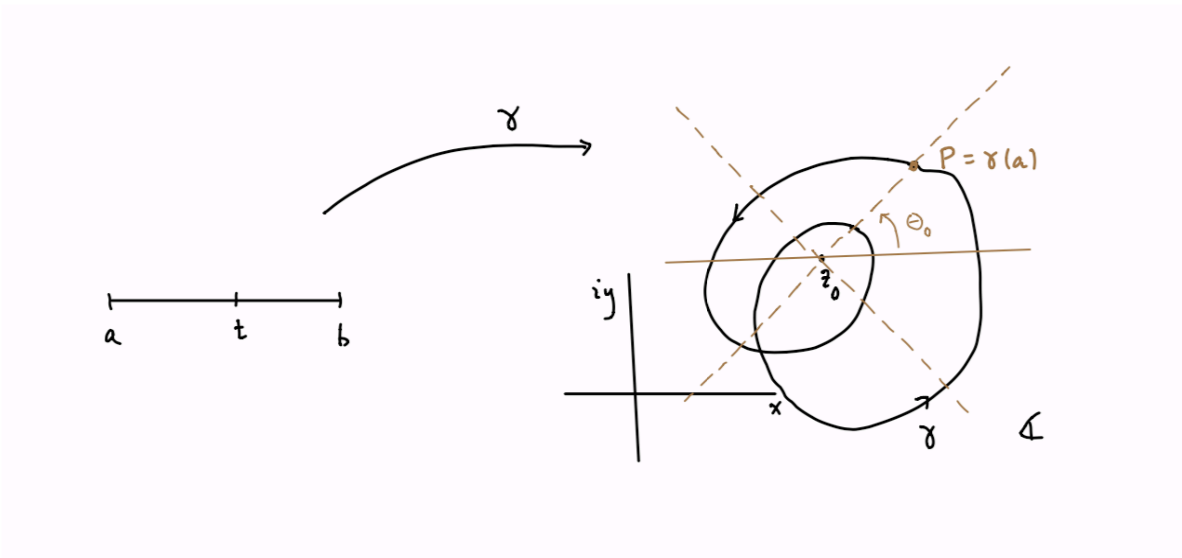
This number is usually easy to compute if you can see the image curve γ.

Example: deduce the winding numbers of the various closed curves above, about the indicated points.

Def We say that z_0 is *inside* γ iff $I(\gamma ; z_0) \neq 0$.

<u>Lemma</u> $I(\gamma\,; z_0)$ is a well defined integer for any continuous closed curve γ with $z_0 \notin \gamma([a,b])$.

proof:



$\gamma([a,b])$ is compact and contains its limit points, so $z_0 \notin \gamma([a,b])$ means γ stays a uniform distance away from z_0, $|z_0 - \gamma(t)| \geq \varepsilon > 0$.

Claim (1): Make a choice θ_0 for the argument of $\gamma(a) - z_0$ (determined up to a multiple of 2π). Then there is a unique way to extend $\theta = \theta(t) = arg\left(\gamma(t) - z_0\right)$ as a continuous function on the interval $[a,b]$, i.e. so that

$$\gamma(t) - z_0 = |\gamma(t) - z_0|\, e^{i\,\theta(t)} \qquad \forall\, t \in [a,b].$$

proof: Consider the open half plane half plane indicated above:

$$H_1 = \left\{ z \in \mathbb{C} \,\middle|\, \theta_0 - \frac{\pi}{2} < arg(z - z_0) < \theta_0 + \frac{\pi}{2} \right\}.$$

As long as $\gamma([a,t]) \subseteq H_1$ there is a unique way to define

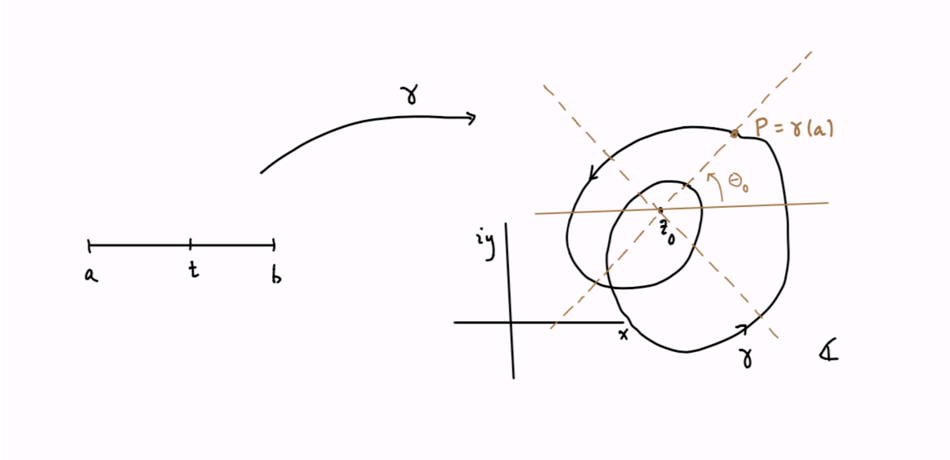
$$\theta(t) = arg\left(\gamma(t) - z_0\right)$$

continously, namely by requiring $\theta_0 - \frac{\pi}{2} < \theta(t) < \theta_0 + \frac{\pi}{2}$ $\theta(t)$.) Let t_1 be the first $t > a$ with

$$\theta_0 - \frac{\pi}{2} = arg\left(\gamma(t_1) - z_0\right) \text{ or } \theta_0 + \frac{\pi}{2} = arg\left(\gamma(t_1) - z_0\right).$$

Then extend $\theta(t)$ for $t > t_1$ using the neighbor halfplane

$$H_2 = \left\{ z \in \mathbb{C} \,\middle|\, arg\left(\gamma(t_1) - z_0\right) - \frac{\pi}{2} < arg\left(z - z_0\right) < arg\left(\gamma(t_1) - z_0\right) + \frac{\pi}{2} \right\}.$$



Continue inductively, finding t_2, t_3 ,.... and half planes H_3, H_4 , ... if necessary. Because $\gamma(t)$ is uniformly continuous and because $\left|\gamma(t) - z_0\right| \geq r$, this process terminates after a finite number of steps with

$$\theta(t) = arg\left(\gamma(t) - z_0\right)$$

defined and continuous on the entire interval $[a, b]$, and so that

$$\gamma(t) - z_0 = \left|\gamma(t) - z_0\right| e^{i\,\theta(t)}.$$

(2) Define $I\left(\gamma\,; z_0\right) := \frac{1}{2\pi}\left(\theta(b) - \theta(a)\right)$. Since $\gamma(a) = \gamma(b)$ and $\theta(b)$ and $\theta(a)$ are argument choices of $\gamma(a) - z_0$, the index is an integer. Any other continuous construction of the argument function, say $\theta_1(t)$ would have

$$\theta_1(t) - \theta(t) = 2\pi\, k(t), \quad k(t) \in \mathbb{Z}.$$

Since $k(t)$ is a difference of continuous functions it is continuous on $[a, b]$ and since it only takes on integer values, i.e. $\theta_1(t) = \theta(t) + 2\pi\, k$, it must be constant. Thus

$$\frac{1}{2\pi}\left(\theta_1(b) - \theta_1(a)\right) = \frac{1}{2\pi}\left(\theta(b) - \theta(a)\right)$$

so $I\left(\gamma\,; z_0\right)$ is well defined.

Theorem If γ, z_0 are as in the preceding discussion, and if γ is also piecewise C^1, then index can be computed with a contour integral:

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\, dz.$$

motivation: locally, in polar coordinates, $z = z_0 + r\,e^{i\theta}$

$$z = z_0 + r\,e^{i\theta}$$

$$dz = (dr)e^{i\theta} + r\,e^{i\theta} i\, d\theta$$

$$\Rightarrow \frac{dz}{z - z_0} = \frac{(dr)e^{i\theta} + r\,e^{i\theta} i\, d\theta}{r\,e^{i\theta}} = \frac{dr}{r} + i\, d\theta.$$

proof: Let $a \le s \le b$, θ_0, $\theta(t)$, H_1, t_1, H_2, t_2 ... as in the earlier discussion. If γ is C^1, then for $a \le s \le t_1$ and using polar coordinates in the intial half plane H_1,

$$\gamma(t) = z_0 + r(t)e^{i\theta(t)}$$

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\gamma'(t)\, dt = \frac{1}{2\pi i}\int_a^s \frac{r'(t)e^{i\theta(t)} + r(t) i\, e^{i\theta(t)}\theta'(t)}{r(t)e^{i\theta(t)}}\, dt$$

$$= \frac{1}{2\pi i}\left(\ln(r(s) - \ln(r(a)) + i(\theta(s) \text{-} \theta(a))\right).$$

Continue for $t_1 \le s \le t_2$, ... and adding more subintervals if γ is only piecewise C^1. Using telescoping series, deduce that for all $a \le s \le b$,

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\gamma'(t)\, dt = \frac{1}{2\pi i}\left(\ln\left(\frac{r(s)}{r(a)}\right) + i(\theta(s) - \theta(a))\right).$$

At $s = b$ for this closed curve $\ln\left(\frac{r(b)}{r(a)}\right) = 0$ so

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\gamma'(t)\, dt = \frac{1}{2\pi}(\theta(b) - \theta(a)) = I(\gamma; z_0).$$

Q.E.D.

Example

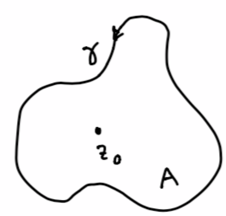
a) Show that for $\gamma(t) = z_0 + r\,e^{it}$, $0 \le t \le 2\,n\pi$, $n \in \mathbb{Z}$ that the winding number of n agrees the contour integral formula

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz.$$

b) Let A be an open connected domain with boundary a p.w. C^1 simple closed curve γ oriented counterclockwise, $z_0 \in A$. Show

$$I(\gamma; z_0) = 1$$

via contour replacement.



The <u>Cauchy Integral Formula</u> (which we will see is amazing in its consequences):

Let $A \subseteq \mathbb{C}$ be open and simply connected
$f: A \to \mathbb{C}$ analytic
$\gamma: [a, b] \to \mathbb{C}$ a piecewise C^1 closed contour in A, $z_0 \notin \gamma([a, b])$.
Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) I(\gamma; z_0) \ .$$

(So, if z_0 is inside γ then $f(z_0)$ is determined and computable just from the values of f along γ !!!)

proof: Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

1) g is analytic in $A \setminus \{z_0\}$ and continuous at z_0.

2) So the modified rectangle lemma holds (see appendix)

3) So the local antiderivative theorem holds for $g(z)$, and therefore the global antiderivative theorem holds, because A is simply connected.

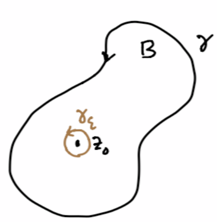
4) So $\int_\gamma g(z)\, dz = 0$ because γ is closed. And since $z_0 \notin \gamma$,

$$0 = \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) \int_\gamma \frac{1}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) 2\pi i\, I(\gamma; z_0)$$

Q.E.D.

<u>Remark</u>: If γ is a counter-clockwise simple closed curve bounding a subdomain B in A, with z_0 inside γ, then the important special case of the Cauchy integral formula can be proven with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$



Appendix: Rectangle lemma improvement:

Previously we proved the

Rectangle Lemma Let $f: D(z_0; r) \to \mathbb{C}$ be analytic. Let $R = [a, b] \times [c, d] \subseteq D(z_0, r)$ be a closed coordinate rectangle inside the disk. (i.e. $R = \{x + iy \mid a \le x \le b, c \le y \le d\} \subseteq D$.) Let $\gamma = \delta R$, oriented counterclockwise. Then

$$\int_\gamma f(z)\, dz = 0.$$

With the rectangle lemma we proved

Local antiderivative Theorem Let $f: D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F: D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.

The proof of the local antiderivative theorem only required that f be continuous, along with the rectangle lemma.

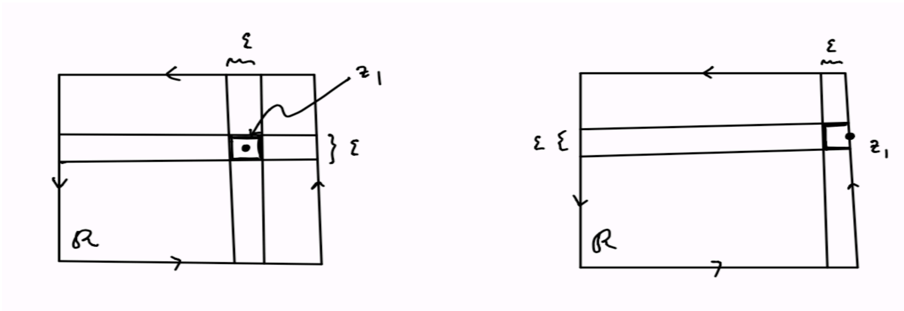
The local antiderivative theorem was used to prove the homotopy lemma, which yielded antiderivatives for continuous functions satisfying the rectangle lemma on all subdiscs, for simply-connected domains. So to deduce that the function $g(z)$ in the previous discussion has a global antiderivative G it suffices to prove:

Theorem The Local antiderivative Theorem also holds if $f: D(z_0; r) \to \mathbb{C}$ is analytic except at a single point z_1 in the disk, where it is only known that f is bounded in a neighborhood of z_1, for example if f is continuous at z_1.

proof: The rectangle lemma + f continuous allows the construction of the antiderivative F. The rectangle lemma used the analyticity of f, but if there's a single point where we don't have analyticity but do have at least that f is bounded, we can still prove that the rectangle lemma holds for all rectangles. Here's now: Let R be chosen.

If $z_1 \notin R$, there's no problem. (Goursat's argument only used subdivision within the rectangle.)

If z_1 is in the interior of R or the boundary of R, subdivide and use a limiting argument with subrectangles and contour integral cancellations, and the boundedness of f near z_1 to deduce the rectangle lemma:



Let $\varepsilon > 0$, subdivide as indicated. Let R_{z_1} be the $\varepsilon \times \varepsilon$ rectangle as indicated above. Apply the rectangle lemma on all other rectangles of the subdivision, note cancellation of contour integrals in the interior of R, and deduce

$$\int_{\delta R} f(z)\, dz = \int_{\delta R_{z_1}} f(z)\, dz.$$

And

$$\left| \int_{\delta R_{z_1}} f(z)\, dz \right| \leq \int_{\delta R_{z_1}} |f(z)|\, |dz| \leq M\, 4\, \varepsilon \to 0 \text{ as } \varepsilon \to 0 ,$$

where M is a local bound on $|f(z)|$ near z_0.

Math 4200-001
Week 7 concepts and homework
2.4
Due Wednesday October 16 at start of class.

2.4 2, 4, 13.

w7.1 Prove the special case of the Cauchy integral formula that we discuss in Friday's notes:

If γ is a counter-clockwise simple closed curve bounding a subdomain B in A, with z_0 inside γ, then the important special case of the Cauchy integral formula can be proven with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$