Math 4200
Wednesday October 30
3.2-3.3 isolated zeroes theorem, uniqueness of analytic extensions; begin 3.3 Laurent series.

Announcements: hw due date was extended to this Friday at 5:00 p.m.
(I'll also accept it now.)

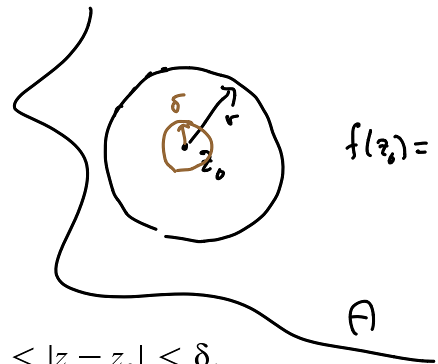
- any questions?

3.2.19 won't be graded.

- next week's HW is at end of today's notes. (It's shorter.)

- If you want to do project instead of final exam, we need to arrange that by next Fri: Nov 8

Consequences of power series for analytic functions:



Theorem (Isolated zeroes theorem). Let

$$\begin{gathered} A \subseteq \mathbb{C} \text{ be an open, connected set,} \\ f: A \to \mathbb{C} \text{ analytic,} \\ D(z_0; r) \subseteq A, \\ f(z_0) = 0. \end{gathered}$$

Then either $\underline{f(z) \equiv 0}$ in $D(z_0; r)$ or $\exists\, \delta > 0$ such that $\underline{f(z) \neq 0}$ $\forall\, \underline{0 < |z - z_0| < \delta}$.

proof: f has convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad |z - z_0| < r.$$

If all the $a_n = 0$ then $f \equiv 0$ in $D(z_0; r)$. Otherwise let a_N be the first non-zero coefficient in the power series, so

$$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n = a_N (z - z_0)^N + a_{N+1} (z - z_0)^{N+1} + \cdots$$

N

and factor out the lowest power ~~of~~ $(z - z_0)$ that appears in this series:

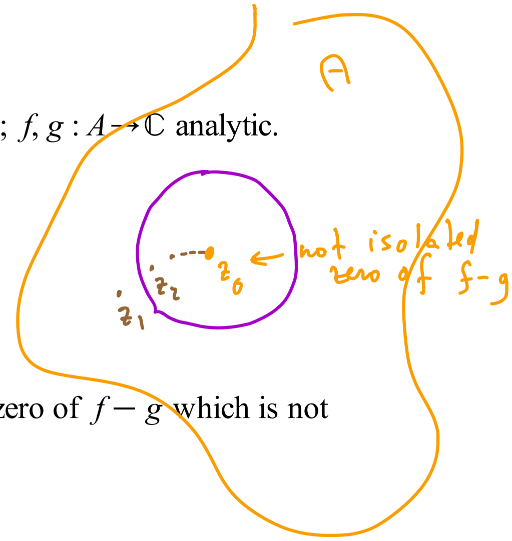
$$f(z) = (z - z_0)^N \left(a_N + \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \right)$$

$$f(z) = (z - z_0)^N g(z)$$

where $g(z_0) = a_N \neq 0$ and $g(z)$ is analytic and hence continous near z_0. Thus there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow g(z) \neq 0$. This proves the claim.

QED.

There is a surprising consequence of the isolated zeroes theorem:



Corollary (Unique extensions theorem) Let $A \subseteq \mathbb{C}$ be open and connected; $f, g : A \to \mathbb{C}$ analytic. Supposed there exists

$$A \subseteq \mathbb{C} \text{ be open and connected,}$$
$$f, g : A \to \mathbb{C} \text{ analytic;}$$
$$\{z_k\} \subseteq A,\ \{z_k\} \to z_0 \in A,\ z_k \neq z_0,\ k \in \mathbb{N}.$$
$$f(z_k) = g(z_k)\ \forall\, k$$

Then $f(z) = g(z)\ \forall\, z \in A$.

proof: $f - g : A \to \mathbb{C}$ is analytic and $(f-g)(z_k) = 0\ \forall\, k$. Thus z_0 is a zero of $f - g$ which is not isolated. Thus by the isolated zeroes theorem,

$$(f-g)(z) \equiv 0,\ \forall\, z \in \mathrm{D}(z_0; r) \subseteq A.$$

(This is already surprising.) Now, consider

$$B := \{z \in A \mid (f-g)^{(n)}(z) = 0\ \forall\, n = 0, 1, 2, \ldots\}.$$

We have $\mathrm{D}(z_0; r) \subseteq B$ since $(f-g)(z) \equiv 0$ in $\mathrm{D}(z_0; r)$.

- B is closed in A because if $\{w_k\} \subseteq B$, $\{w_k\} \to w \in A$ then
$0 = (f-g)^{(n)}(w_k) \to (f-g)^{(n)}(w)\ \forall\, n$.

- B is open in A because if $z_1 \in B$ the Taylor series for f at z_1 is the zero function, so for any
$\mathrm{D}(z_1; r) \subseteq A$ we also have $\mathrm{D}(z_1; r) \subseteq B$.

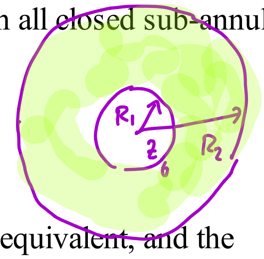
Thus, since A is connected, $B = A$.

Example (also see one of your homework problems). It is not clear without a lot of work why the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be extended as an analytic function (with different formulas), beyond the plane $\mathrm{Re}(s) > 1$ on which the series converges. But in fact, it can be extended as an analytic function on $\mathbb{C} \setminus \{1\}$. The Unique extensions theorem says there's only one possible extension.

3.3 Laurent series. If $f(z)$ is an analytic function on a punctured disk or on an annulus centered at z_0, then f can be expressed as a power series expansion using non-negative *and* negative powers of $(z - z_0)$, and this series converges in the open annulus and uniformly absolutely in all closed sub-annuli. These expansions are called *Laurent series*.



Laurent Series Theorem For $0 \le R_1 < R_2$ let

$$A = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f : A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}.$$
$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$.
And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \ge r_1 > R_1$.

(3) The Laurent coefficients a_n, b_m are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$
$$b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta) (\zeta - z_0)^{m-1} d\zeta.$$

In particular the contour integral of f itself has value

$$\int_\gamma f(\zeta) d\zeta = 2\pi i\, b_1.$$

For this reason, the coefficient b_1 of $\frac{1}{z - z_0}$ in the Laurent series, is called the *residue* of f at z_0.

(Because it's the only part of the Laurent series you need to know in order to compute the contour integral of f.)

Note: (2) $\Rightarrow$ (1) of the theorem is immediate, since uniform limits of analytic functions are analytic. We'll prove (2) $\Rightarrow$ (3) in today's notes, and then do (1) $\Rightarrow$ (2) on Friday.

Examples:

1) Consider

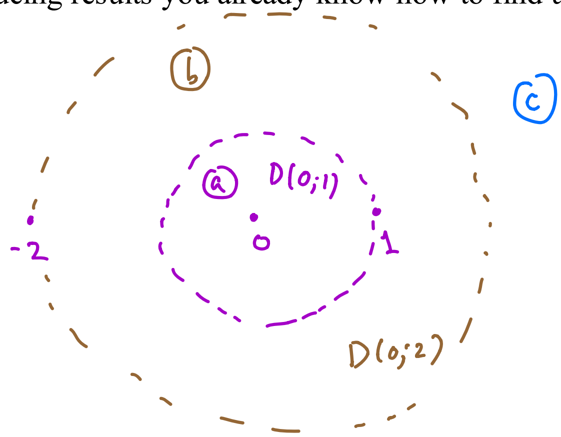
$$f(z) = \frac{1}{(z-1)(z+2)} = \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).$$

Find the following series expansions for $z_0 = 0$:

a) Taylor series for $|z| < 1$
b) Laurent series for $1 < |z| < 2$.
c) Laurent series for $|z| > 2$.

in all of these, we use $\displaystyle\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad |w|<1$

Use residues from the Laurent series to compute $\displaystyle\int_\gamma f(z)\,dz$ for the three index-one circles centered at the origin, of radii $\frac{1}{2}, \frac{3}{2}, 3$. Notice that this is reproducing results you already know how to find using the Cauchy integral formula and other means.



a) $|z|<1$.

$$\frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right)$$

1, −2 are big moduli

$$= \frac{1}{3}\left(\frac{1}{-1(1-z)} - \frac{1}{2(1+\frac{z}{2})}\right)$$

$(1-(-\frac{z}{2}))$

$$= -\frac{1}{3}\sum_{n=0}^{\infty} z^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

$$\underbrace{R=1 \qquad\qquad R=2}_{R=1}$$

b) $1<|z|<2$: $\displaystyle\frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right) = \frac{1}{3}\left(\frac{1}{z(1-\frac{1}{z})} - \frac{1}{2(1-(-\frac{z}{2}))}\right)$

z big mod 2 big mod.

$$= \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

conv for $|\frac{1}{z}|<1$, $|z|>1$ conv. for $|\frac{z}{2}|<1$, $|z|<2$.

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

$$\boxed{= \frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{z^m} - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n}$$

for $1<|z|<2$

c) $|z|>2$

$$\frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right)$$

"z" big modulus.

$$= \frac{1}{3}\left(\frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-(-\frac{2}{z}))}\right)$$

$$= \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{3z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n}$$

$|z|>1$ $|z|>2$.

2a) What is the Laurent series for $z\, e^{\frac{1}{z}}$ in $\mathbb{C} \setminus \{0\}$?
2b) What is the value of

$$\int_\gamma z\, e^{\frac{1}{z}}\, dz$$

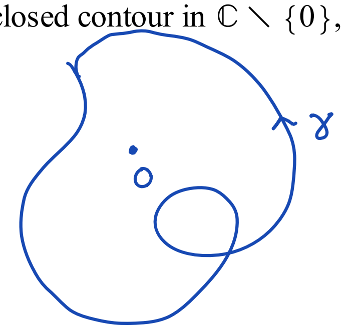
if γ is a closed contour in $\mathbb{C} \setminus \{0\}$, with $I(\gamma; 0) = 1$?

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

$$= z \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^n}\right)$$

$$= z\left(1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \cdots\right)$$

$$= \underbrace{z + 1}_{S_1(z)} + \underbrace{\sum_{m=1}^{\infty} \frac{1}{(m+1)!} \frac{1}{z^m}}_{S_2(z)}.$$



$$\int_\gamma z e^{\frac{1}{z}}\, dz = \int_\gamma z + 1 + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots\, dz$$

converges uniformly on γ

exchange inf sum & integral

$$= \int_\gamma z\, dz + \int_\gamma 1\, dz + \int_\gamma \frac{1}{2}\frac{1}{z}\, dz + \int_\gamma \frac{1}{3!}\frac{1}{z^2}\, dz$$

$$= \underset{\text{FTC or Cauchy}}{0} + \underset{\text{FTC or Cauchy}}{0} + \cdots \quad \text{FTC}$$

$$+ \frac{1}{2}\, 2\pi i\, \underbrace{I(\gamma; 0)}_{1} + 0 + \cdots$$

$$= \pi i$$

$$\int_\gamma \frac{1}{z^2}\, dz = -\frac{1}{z}\Big]_{\text{init pt}}^{\text{term pt}} = 0$$

$$f(z) = \frac{1}{z^2} \qquad F(z) = -\frac{1}{z}$$

$$\left(\int_\gamma z^n\, dz = \frac{z^{n+1}}{n+1}\Big]_{\text{init pt}}^{\text{term pt}} = 0 \quad \forall n \in \mathbb{Z} \text{ except } n = -1\right)$$

to note.
residue of $ze^{1/z}$ at $z_0 = 0$ is defined as the coeff of $\frac{1}{z - z_0} = \frac{1}{z}$ in Laurent series.

contour integral was just $2\pi i$ times the residue *proof of (2)* $\Rightarrow$ *(3)* in the Laurent series theorem: Friday!

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}$$
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$$b_m = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) (\zeta - z_0)^{m-1} d\zeta.$$

In particular the contour integral of f itself has value

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i\, b_1.$$

proof: We'll write $f(\zeta) = S_1(\zeta) + S_2(\zeta)$ and just compute the contour integrals above. We'll use the uniform convergence of the series $S_1(\zeta), S_2(\zeta)$ on γ to interchange the integrals with the summations:

Math 4200-001
Week 10 concepts and homework
3.2-3.3
Due Wednesday November 6 at start of class.

3.3 1ab, 4, 6, 8, 13, 15, 17, 18, 19, 20

w10.1 Let f be an entire function. Suppose $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers n. Is it possible for $f(-1)$ to equal -1? Explain.

w10.2) Use power series or L'Hopital's rule to find

$$\lim_{z \to 0} \frac{\cos(z) - 1}{z \sin(z)}$$

w10.3) Continuing the text problem 3.3.4, find the Laurent series for

$$\frac{1}{z(z-1)(z-2)}$$

valid for $|z| > 2$.

w10.4) Which of these functions has a removable singularity at $z = 0$?

a) $\dfrac{\cos(z) - 1}{z \sin(z)}$ (see w10.2)

b) $\dfrac{\cos(z) - 1}{z^3}$.