Math 4200
Wednesday October 30
3.2-3.3 isolated zeroes theorem, uniqueness of analytic extensions; begin 3.3 Laurent series.

Announcements:

Consequences of power series for analytic functions:

<u>Theorem</u> (Isolated zeroes theorem). Let

$A \subseteq \mathbb{C}$ be an open, connected set,
$f: A \to \mathbb{C}$ analytic,
$D(z_0; r) \subseteq A$,
$f(z_0) = 0.$

Then either $f(z) \equiv 0$ in $D(z_0; r)$ or $\exists\, \delta > 0$ such that $f(z) \neq 0 \quad \forall\, 0 < |z - z_0| < \delta.$

proof: f has convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad |z - z_0| < r.$$

If all the $a_n = 0$ then $f \equiv 0$ in $D(z_0; r)$. Otherwise let a_N be the first non-zero coefficient in the power series, so

$$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n$$

and factor out the lowest power of $(z - z_0)$ that appears in this series:

$$f(z) = (z - z_0)^N \left(a_N + \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \right)$$

$$f(z) = (z - z_0)^N g(z)$$

where $g(z_0) = a_N \neq 0$ and $g(z)$ is analytic and hence continous near z_0. Thus there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow g(z) \neq 0$. This proves the claim.

QED.

There is a surprising consequence of the isolated zeroes theorem:

Corollary (Unique extensions theorem) Let $A \subseteq \mathbb{C}$ be open and connected; $f, g : A \to \mathbb{C}$ analytic.
Supposed there exists

$$A \subseteq \mathbb{C} \text{ be open and connected,}$$
$$f, g : A \to \mathbb{C} \text{ analytic;}$$
$$\{z_k\} \subseteq A,\ \{z_k\} \to z_0 \in A,\ z_k \neq z_0,\ k \in \mathbb{N}.$$
$$f(z_k) = g(z_k)\ \forall\ k$$

Then $f(z) = g(z)\ \forall\ z \in A$.
proof: $f - g : A \to \mathbb{C}$ is analytic and $(f-g)(z_k) = 0\ \forall\ k$. Thus z_0 is a zero of $f-g$ which is not isolated. Thus by the isolated zeroes theorem,

$$(f-g)(z) \equiv 0,\ \forall\ z \in D(z_0; r) \subseteq A.$$

(This is already surprising.) Now, consider

$$B := \{z \in A \mid (f-g)^{(n)}(z) = 0\ \forall\ n = 0, 1, 2, \ldots\}.$$

We have $D(z_0; r) \subseteq B$ since $(f-g)(z) \equiv 0$ in $D(z_0; r)$.

- B is closed in A because if $\{w_k\} \subseteq B$, $\{w_k\} \to w \in A$ then $0 = (f-g)^{(n)}(w_k) \to (f-g)^{(n)}(w)\ \forall\ n$.

- B is open in A because if $z_1 \in B$ the Taylor series for f at z_1 is the zero function, so for any $D(z_1; r) \subseteq A$ we also have $D(z_1; r) \subseteq B$.

Thus, since A is connected, $B = A$.

Example (also see one of your homework problems). It is not clear without a lot of work why the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be extended as an analytic function (with different formulas), beyond the plane $\mathrm{Re}(s) > 1$ on which the series converges. But in fact, it can be extended as an analytic function on $\mathbb{C} \setminus \{1\}$. The Unique extensions theorem says there's only one possible extension.

3.3 Laurent series. If $f(z)$ is an analytic function on a punctured disk or on an annulus centered at z_0, then f can be expressed as a power series expansion using non-negative *and* negative powers of $(z - z_0)$, and this series converges in the open annulus and uniformly absolutely in all closed sub-annuli. These expansions are called *Laurent series.*

Laurent Series Theorem For $0 \le R_1 < R_2$ let

$$A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f: A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}.$$

$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$.
And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \ge r_1 > R_1$.

(3) The Laurent coefficients a_n, b_m are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{m-1} d\zeta.$$

In particular the contour integral of f itself has value

$$\int_\gamma f(\zeta) d\zeta = 2\pi i\, b_1.$$

For this reason, the coefficient b_1 of $\frac{1}{z - z_0}$ in the Laurent series, is called the *residue* of f at z_0.

(Because it's the only part of the Laurent series you need to know in order to compute the contour integral of f.)

Note: (2) $\Rightarrow$ (1) of the theorem is immediate, since uniform limits of analytic functions are analytic. We'll prove (2) $\Rightarrow$ (3) in today's notes, and then do (1) $\Rightarrow$ (2) on Friday.

Examples:

1) Consider

$$f(z) = \frac{1}{(z-1)(z+2)} = \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).$$

Find the following series expansions for $z_0 = 0$:

a) Taylor series for $|z| < 1$
b) Laurent series for $1 < |z| < 2$.
c) Laurent series for $|z| > 2$.

Use residues from the Laurent series to compute $\int_\gamma f(z)\, dz$ for the three index-one circles centered at the origin, of radii $\frac{1}{2}, \frac{3}{2}, 3$. Notice that this is reproducing results you already know how to find using the Cauchy integral formula and other means.

2a) What is the Laurent series for $z\,e^{\frac{1}{z}}$ in $\mathbb{C} \setminus \{0\}$?
2b) What is the value of

$$\int_{\gamma} z\,e^{\frac{1}{z}}\,dz$$

if γ is a closed contour in $\mathbb{C} \setminus \{0\}$, with $I(\gamma; 0) = 1$?

proof of (2) $\Rightarrow$ *(3)* in the Laurent series theorem:

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z-z_0)$:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n \quad + \sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}$$

$$:= S_1(z)+S_2(z).$$

Here $S_1(z)$ converges for $|z-z_0|<R_2$ and uniformly absolutely for $|z-z_0|\le r_2<R_2$.
And $S_2(z)$ converges for $|z-z_0|>R_1$, and uniformly for $|z-z_0|\ge r_1>R_1$.

(3) The Laurent coefficients a_n, b_m are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0)=1$, e.g. any circle of radius r, with $R_1<r<R_2$, then

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$$b_m=\frac{1}{2\pi i}\int_{\gamma}f(\zeta)(\zeta-z_0)^{m-1}d\zeta.$$

In particular the contour integral of f itself has value

$$\int_{\gamma}f(\zeta)d\zeta=2\pi i\, b_1.$$

proof: We'll write $f(\zeta)=S_1(\zeta)+S_2(\zeta)$ and just compute the contour integrals above. We'll use the uniform convergence of the series $S_1(\zeta), S_2(\zeta)$ on γ to interchange the integrals with the summations:

Math 4200-001
Week 10 concepts and homework
3.2-3.3
Due Wednesday November 6 at start of class.

3.3 1ab, 4, 6, 8, 13, 15, 17, 18, 19, 20

w10.1 Let f be an entire function. Suppose $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers n. Is it possible for $f(-1)$ to equal -1? Explain.

w10.2) Use power series or L'Hopital's rule to find

$$\lim_{z \to 0} \frac{\cos(z) - 1}{z \sin(z)}$$

w10.3) Continuing the text problem 3.3.4, find the Laurent series for

$$\frac{1}{z(z-1)(z-2)}$$

valid for $|z| > 2$.

w10.4) Which of these functions has a removable singularity at $z = 0$?

a) $\dfrac{\cos(z) - 1}{z \sin(z)}$ (see w10.2)

b) $\dfrac{\cos(z) - 1}{z^3}$.