Math 4200
Monday October 28
3.2 Finish Taylor series/power series facts and examples.

Announcements: HW questions? ... there are quite a few this week

3.1.14. $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ analytic in $|z|<1$ & $|z|>1$.

case 1 in $|z|\leq r<1$, $\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{\cancel{|z|^n}\, r^n}{1-r^{2n}} \leq M r^n$

$|1+z^{2n}| \geq 1-|z|^{2n}$ $\quad |b-a| \geq |b|-|a|$

$\geq 1-r^{2n}$

$\frac{1}{1-r^{2n}} \leq M$

outside unit disk case 2 $R \geq |z| \geq r > 1$

$\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{|z|^n}{}$

$|1+z^{2n}| \geq |z|^{2n}-1 \ldots$

3.2.4. Taylor series for

$\cos z, \sin z, (1+z)^{\alpha}$ @ $z_0=0$

entire series & rad of conv.

$\frac{1}{2}(e^{iz}+e^{-iz})$

use Taylor for e^w, $w=iz$, $w=-iz$

use Taylor, use today's ideas for radius of conv.

Fact Review:

1) Every *power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad\qquad a_n, z_0 \in \mathbb{C}.$$

has a unique radius of convergence $R \in [0, \infty]$ such that the power series above converges $\forall\, z$ with $|z - z_0| < R$ (and converges absolutely for any $r < R$), and diverges for all z with $|z - z_0| > R$. The limit is an analytic function.

2) Power series may be differentiated and integrated term by term to get derivatives and antiderivatives of f, and the resulting power series have the same radius of convergence as f.

3) Therefore, whenever $R > 0$, after differentiating k times and substituting $z = z_0$ into the power series, one realizes that the power series is actually the Taylor series for f,

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

As a consequence, power series that yield the same analytic function in a neighborhood of z_0 must be identical, because they are the Taylor series for that function, centered at z_0. (We called this Theorem A on Friday)

4) Conversely, given a function $f \colon D(z_0, r) \to \mathbb{C}$ which is analytic, the Taylor series power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

is guaranteed to converge in $D(z_0, r)$. (To be shown today.) (We called this Theorem B on Friday.) Furthermore, if the Taylor series converges to f in some $D(z_0, R_1)$ and f is also unbounded in $D(z_0, R_1)$ (as $|z| \to R_1$), then R_1 *is* the radius of convergence for the power series. (To be reviewed today.)

$R \geq R_1$ since T.S. converges on $D(z_0; R_1)$. (to f)

but if R was $> R_1$ that would imply on $\overline{D(z_0, R_1)} \subset D(z_0, R)$, f is cont, so it's bounded.

$\Rightarrow\!\Leftarrow$ to f unbounded on $D(z_0; R_1)$

Unfinished examples from Friday:

1) Find the Taylor series for $f(z) = \log(1+z)$ at $z_0 = 0$, along with its radius of convergence.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad R=1$$

$$\Rightarrow \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \qquad R=1$$

must be the Taylor ~~power~~ series by uniqueness of power series

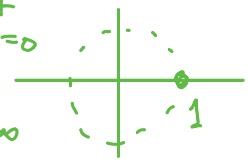
$$\int: \quad \log(1+z) = \log 1 + \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} \qquad R=1.$$

$$\left(= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^m}{m} \right)$$

$m = n+1$

compare to Thm B

nearest pt to $z_0=0$ where $\frac{1}{1-z} \to \infty$



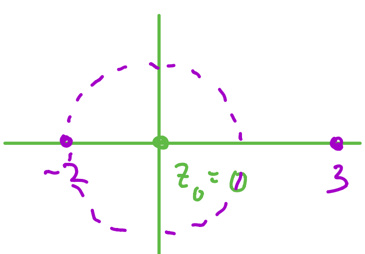
$R=1$ ✓

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{(z-z_0)^{n+1}}{n+1} + F(z_0)$$

2) Find the Taylor series of $f(z) = \dfrac{1}{z^2 - z - 6} = \dfrac{1}{5}\left(\dfrac{1}{z-3} - \dfrac{1}{z+2}\right)$ at $z_0 = 0$, along with its radius of convergence.

Theorem B says we can find R before we even know the series



$f \to \infty$ at $z = -2, 3$

$\Rightarrow R = 2.$

$$|z|<2 \qquad f(z) = \frac{1}{5}\left(\frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{2\left(1-\left(-\frac{z}{2}\right)\right)} \right)$$

going to use geometric series!

$$= \frac{1}{-15}\,\frac{1}{1-z/3} - \frac{1}{10}\,\frac{1}{1-\left(-\frac{z}{2}\right)}$$

$\uparrow |z|<3$ $\qquad \uparrow |z|<2$

$$= -\frac{1}{15}\sum_{n=0}^{\infty} \frac{1}{3^n} z^n - \frac{1}{10}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n$$

unnecessary

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{15}\frac{1}{3^n} - \frac{1}{10}\frac{1}{2^n} \right) z^n$$

that's the power series & $R=2$

3) Define $\log(z) = \ln|z| + i\,arg(z)$ on the branch domain $0 < arg(z) < 2\pi$. Find the Taylor series for $\log(z)$ at $z_0 = 1 + i$, and find the radius of convergence using the ratio test for absolute convergence. Explain why your answer may seem surprising at first.

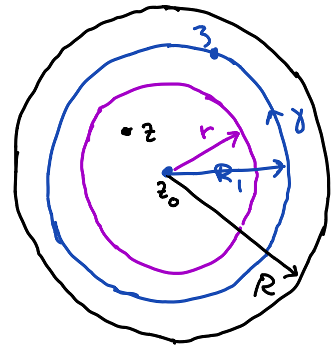
(skip)

Theorem B If f is analytic in $D(z_0;R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges to f in $D(z_0;R)$. Thus the radius of convergence of the power series is at least R.

proof: Let $|z-z_0| \le r < R_1 < R$, $\gamma(t) = z_0 + R_1 e^{it}$, $0 \le t \le 2\pi$, the circle $\zeta - z_0 = R_1$.
Then the Cauchy integral formula reads



$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We use geometric series magic:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)}\frac{1}{1 - \frac{(z-z_0)}{(\zeta - z_0)}}\, d\zeta$$

R $\quad \left|\frac{z-z_0}{\zeta - z_0}\right| \le \frac{r}{R_1} < 1$

using the geometric series for $\dfrac{1}{1-w}$ with $|w| \le \dfrac{r}{R_1}$:

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta - z_0}\right)^n d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} f(\zeta)\frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Because $|f|$ is bounded on γ and

$$\frac{|z-z_0|^n}{|\zeta - z_0|^{n+1}} \le \frac{1}{R_1}\left(\frac{r}{R_1}\right)^n,$$

the series which is the integrand converges uniformly on γ so we may interchange the summation with the integration, (and then pull each $(z-z_0)^n$ through the integral:

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_\gamma \sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty}(z-z_0)^n \left(\frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right)$$

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int \frac{\cancel{n!}\, f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

by the Cauchy integral formula for derivatives!

Q.E.D.

Sometimes it is useful to know you can multiply power series term by term, and without having to worry about radius of convergence issues. This theorem makes it a breeze:

<u>Theorem C</u> (Multiplying power series): Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n == b_0 + b_1 (z - z_0) + b_2 (z - z_0)^2 + \ldots$$

in $D(z_0; R)$. Then the power series for $f(z)g(z)$ also converges in $D(z_0; R)$ and is given by

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)(z - z_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(z - z_0)^2 + \ldots.$$

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) (z - z_0)^n,$$

in other words, what you expect by formally multiplying and collecting all coefficients for each $(z - z_0)^n$.

proof: We know that power series are Taylor series. Therefore,

$$f(z)g(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(z_0)}{n!} (z - z_0)^n$$

will converge in $D(z_0; R)$. Compute the various derivatives, using the product rule for first, second, ..., n^{th} derivatives of product functions (via induction and the binomial theorem).

$$(fg)(z_0) = a_0 b_0$$

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) = a_1 b_0 + a_0 b_1$$

$$(fg)''(z_0) = f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0)$$

$$(fg)''(z_0) = (2a_2)b_0 + 2a_1 b_1 + a_0(2b_2) = 2!(a_2 b_0 + a_1 b_1 + a_0 b_2)$$

In general and using the product rule, (checked by induction, as in proof of binomial theorem in first HW),

$$\frac{(fg)^{(n)}}{n!}(z_0) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(z_0) g^{(n-j)}(z_0)$$

$$= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (j!\, a_j)\ (n-j)!\, b_{n-j} = \frac{\cancel{n!}}{\cancel{n!}} \sum_{j=0}^{n} a_j b_{n-j}$$

Q.E.D.

Example: Find the first three non-zero terms in the Taylor series for $\underbrace{\sec(z) = \frac{1}{\cos(z)}}$ at $z_0 = 0$. Hint, rewrite the equation as

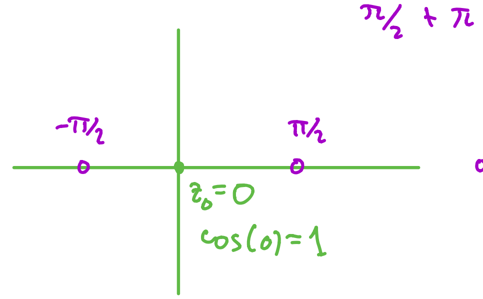
$$\cos(z)\,\sec(z) = 1.$$

what is the radius of convergence of this power series?

singularities for sec z

$\pi/2 + \pi k$

$$\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + \cdots\right)$$



$$= \underbrace{1}_{\text{Taylor series for 1.}}$$

$R = \pi/2$

match coeffs (Thm C)

z^0: $b_0 = 1$

z^1: $b_1 = 0$

z^2: $b_2 - \frac{b_0}{2} = 0 \Rightarrow b_2 = \frac{1}{2} b_0 = \frac{1}{2}$

z^3: $b_3 - \frac{b_1}{2} = 0 \Rightarrow b_3 = 0$

z^4: $-\frac{b_2}{2!} + \frac{b_0}{4!} + b_4 = 0 \Rightarrow -\frac{1}{4} + \frac{1}{4!} = -b_4$

$b_4 = \frac{1}{4} - \frac{1}{4!} = --$