So seq. is Cauchy.

$= L < \infty$. so $\forall \varepsilon > 0 \ \exists N$ s.t. $m < n$, $m, n \geq N$ $\Rightarrow \sum_{j=m+1}^{n} |a_j| < \varepsilon$

Def The series $\sum_{j=1}^{\infty} a_j$ converges *absolutely* if and only if $\sum_{j=1}^{\infty} |a_j| < \infty$.

Theorem: Absolute convergence implies convergence .

proof:

we'll check orig. seq. is Cauchy.

Let $\varepsilon > 0$. $\exists N$ s.t. $m < n$, $m, n \geq N \overset{?}{\Rightarrow} \underbrace{|S_n - S_m|}_{\left|\sum_{j=m+1}^{n} a_j\right| \leq \sum_{j=m+1}^{n} |a_j| < \varepsilon} < \varepsilon$

↑ use that N

There is a useful test for uniform convergence of a series of functions on a domain A - namely a uniform absolute convergence test. It's called the *Weierstrass M test* (maybe M is chosen because of the word M*odulus*):

(g_n cont, analytic)

Theorem C (Weierstrass M test) Let $\{g_n(z)\}$, $g_n : A \to \mathbb{C}$. If

$$\forall\, n \in \mathbb{N}\ \exists\, M_n \in \mathbb{R} \text{ such that}$$

$$|g_n(z)| \leq M_n\ \forall\, z \in A$$

and if

$$\sum_{n=1}^{\infty} M_n < \infty .$$

then $\sum_{j=1}^{\infty} g_j(z)$ converges uniformly on A. (And in this case, if each g_n is analytic, so is $g(z) = \sum_{j=1}^{\infty} g_j(z)$.)

proof:

Need to show uniform Cauchy for partial sums.

Let $\varepsilon > 0$. Pick N s.t. $m, n \geq N$, $m < n$

$$\Rightarrow \sum_{j=m+1}^{n} M_j < \varepsilon.$$

then $|S_n(z) - S_m(z)| = \left|\sum_{j=m+1}^{n} g_j(z)\right| \leq \sum | \ | \leq \sum_{j=m+1}^{n} M_j < \varepsilon.$

$(\forall z \in A)$

∎

Examples

(1) Last week we discussed the Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which converges uniformly absolutely for $\mathrm{Re}(s) > 1 + \varepsilon$, for each positive ε, so is analytic for $\mathrm{Re}(s) > 1$.

✓

(2) Consider ~~Show that~~

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad |z|<1. \checkmark$$

Use the Weierstrass M test to show this series converges uniformly on $D(0, r)$ for any $r < 1$, so converges to an analytic function in all of $D(0; 1)$. What analytic function is this? (By the way, this is the most important power series in Complex Analysis. :-))

$$|z| \le r \Rightarrow \sum_{n=0}^{\infty} \underbrace{|z^n|}_{|z|^n} = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for } r<1$$

(you know this

$(M_n = r^n$ in W.M. test).

$$S_n(z) = 1 + z + z^2 + \cdots + z^n$$

$$\left.\begin{aligned} S_{n+1}(z) &= S_n + z^{n+1} \\ S_{n+1}(z) &= z S_n(z) + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} S_n + z^{n+1} &= z S_n + 1 \\ S_n(1-z) &= 1 - z^{n+1} \\ S_n &= \frac{1-z^{n+1}}{1-z} \end{aligned}$$

(or do long div $\frac{z^{n+1}-1}{z-1} = 1+z+\cdots+z^{n-1}$?)

$$\text{if } |z|<1, \lim_{n\to\infty} S_n = \frac{1-0}{1-z} = \boxed{\frac{1}{1-z}}$$

(3) Show that the series $\sum_{n=0}^{\infty} z^n$ diverges for all $|z| \ge 1$.

$$\text{if } \sum_{n=0}^{\infty} a_n \text{ converges, then } \underbrace{S_{n+1} - S_n}_{\downarrow \; L-L=0} = a_n$$

$$\text{so } \{|a_n|\} \to 0.$$

$$|z| \ge 1 \text{ then } |z|^n \ge 1 \not\to 0.$$

(4) Show that the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots.$$

converges uniformly for $|z| \le R$, so converges to an analytic function $\forall\, z$. Then use the term by term differentiation theorem to show that $f'(z) = f(z)$ and use this to identify $f(z)$.

Calc.

$$|z| \le R. \quad \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \le \sum_{n=0}^{\infty} \frac{R^n}{n!} = e^R < \infty$$

$$\text{or use ratio test} : \left(\sum_{j=0}^{\infty} a_j \,, \quad \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| < 1 \Rightarrow \text{abs conv.} \right)$$

$|z| \le R.$

$$\cancel{\lim_{n\to\infty} \frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^n} = \lim_{n\to\infty} \frac{|z|}{n+1} \le \lim_{n\to\infty} \frac{R}{n+1} = 0}$$

$$|z| \le R \Rightarrow \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \le \sum_{n=0}^{\infty} \frac{R^n}{n!} \quad \leftarrow \text{use ratio test for this}$$

$$\frac{R^{n+1}}{(n+1)!} \frac{n!}{R^n} = \frac{R}{n+1} \to 0.$$

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$f'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{\substack{m=0 \\ m=n-1}}^{\infty} \frac{z^m}{m!} = f(z).$$

$$f'(z) = f(z)$$

$$\Rightarrow \quad f'(z) - f(z) = 0$$

$$(2280!) \qquad e^{-z} \left[f'(z) - f(z) \right] = 0$$

$$\frac{d}{dz} \left(e^{-z} f(z) \right) = 0$$

$$\Rightarrow e^{-z} f(z) = C$$

$$@\ z = 0 : \ 1 \cdot 1 = C$$

$$\Rightarrow f(z) = 1 \cdot e^{z} \quad !$$

Math 4200
Friday October 25
3.2 Power series and Taylor series for analytic functions. We'll begin with the examples of power series at the end of Wednesday's notes

Announcements:

- you were right → the Weirstrass M test is in 3210 text (for fcns of $x \in \mathbb{R}$)
- also power series, but we'll have a Theorem today you haven't seen before. (Do you recall having to use mean value theorem error estimates to prove convergence of Taylor series to fcns? no more!!)

Power series

Consider the *power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C}.$$

Theorem:
(1) There exists unique $R \in [0, \infty]$ such that the power series above converges $\forall\, z$ with $|z - z_0| < R$ and diverges for all z with $|z - z_0| > R$. This value of R is called the *radius of convergence* of the power series.

(2) For $r < R$, the convergence of the power series is uniform $\forall\, z \in D(z_0, r)$. Thus f is analytic in $D(z_0; R)$.

(3) Furthermore, $f'(z)$ can be computed via term by term differentiation; and the antiderivatives $F(z)$ of $f(z)$ can be computed by term by term antidifferentiation. The power series for f' and F have the same radius of convergence R as does the power series f.

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad \forall\, z \in D(z_0, R)$$

$$F(z) = F(z_0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} \qquad \forall\, z \in D(z_0, R)$$

proof: If the radius of convergence exists, it must be unique. Define the potentially different number $R := sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n| r^n < \infty\}$. We'll show that this number R satisfies the two required convergence conditions to be the radius of convergence, so it *is* the radius of convergence.

$0 \leq R \leq \infty$

(i) Let $r < R$ and apply the Weirestrass M test in $\overline{D}(z_0, r)$ to deduce uniform convergence of the power series for $f(z)$ in $D(z_0, r)$. Thus the power series converges in $D(z_0, R)$ to an analytic function and we have shown (2), and half of (1).

choose R_1 $\quad r < R_1 < R$

$\underline{\sum |a_n| R_1^n < \infty}$ by def of R. So $|a_n| R_1^n \to 0$ as $n \to \infty$
So $\exists\, C$ s.t. $|a_n| R_1^n \leq C$.

use W.M. test on $\overline{D(z_0, r)}$.

$$\sum_{n=0}^{\infty} |a_n| |z - z_0|^n \leq \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} |a_n| R_1^n \left(\frac{r}{R_1}\right)^n$$

$$\leq \sum_{n=0}^{\infty} C \left(\frac{r}{R_1}\right)^n = C\left(\frac{1}{1 - \frac{r}{R_1}}\right) < \infty.$$

$\frac{r}{R_1} < 1$

(ii) Let $|z_1 - z_0| = R_1 > R$. If $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges then the sequence of partial sums is Cauchy, so the n^{th} terms of the series, $a_n (z_1 - z_0)^n = S_{n+1} - S_n$, must converge to zero, so there is a uniform bound C on the modulus of all the terms $a_n (z_1 - z_0)^n$ in the series. Use this to show that

$|a_n| R_1^n \to 0$

$$r < R_1 \qquad \sum_{n=1}^{\infty} |a_n| r^n < \infty \qquad \forall\, r < R_1$$

which would contradict the *sup* definition of R. Thus the power series for $f(z)$ diverges whenever $|z - z_0| > R$, and have finished showing (1).

$$r < R_1 \quad \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} \underbrace{|a_n| R_1^n}_{\le C} \left(\frac{r}{R_1}\right)^n < \infty.$$

(3) It's possible $R = 0$ in which case the power series only converges at $z = z_0$. Otherwise $R > 0$, and since the power series for $f(z)$ converges in $D(z_0, R)$, and uniformly for any closed subdisk (concentric or not, since each closed subdisk is contained in a closed concentric sub-disk), we deduce from Theorem B′ Wednesday that the term-by-term differentiated power series for $f'(z)$ also converges in $D(z_0, R)$, and uniformly for any closed subdisk. Thus the radius of convergence for the $f'(z)$ series is at least the radius of convergence for the f series. But using the alternate characterization of radius of convergence from part (i), the radius convergence for the series for $f'(z)$ is at most R, since the moduli of the terms in the f' series are larger than in the f series:

$$\text{rad of conv for } f' = sup\{r \geq 0 \mid \tfrac{1}{r}\sum_{n=1}^{\infty} n|a_n| r^n < \infty\} \leq sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n| r^n < \infty\} = R. = \text{radius of conv for } f$$

Thus the radii of convergence for f, f' must be the same. Similarly, the radius of convergence for F must be at least $R > 0$ because $\frac{|a_n|}{n+1} \leq |a_n|$. But the radius of convergence for F and $F' = f$ are equal by our earlier reasoning, so the radius of convergence of F is also R.

QED.

Theorem A. (Uniqueness of power series representations) If f is given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C} \qquad a_n, z_0 \in \mathbb{C}$$

with positive radius of convergence R then

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n = 0, 1, 2, ...$$

so the power series for f centered at z_0 *is* the Taylor series for f centered at z_0. In particular, a given analytic function whose domain of analyticity includes z_0 can have only one power series representation centered at z_0.

proof: We know from the previous Theorem that we have

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad \forall\, z \in \mathrm{D}(z_0, R)$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}$$

and inductively, for $k \in \mathbb{N}$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k} \qquad \forall\, z \in \mathrm{D}(z_0, R),\ \forall\, k \in \mathbb{N}.$$

evaluating at z_0 only the first term in the series is nonzero, so

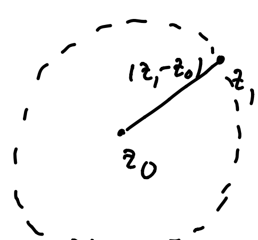
$$f^{(k)}(z_0) = k!\, a_k \quad \Rightarrow a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Theorem B If f is analytic in $\mathrm{D}(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $\mathrm{D}(z_0; R)$. Thus the radius of convergence of the Taylor series is at least R. Thus if $\exists\, z_1$ such that $\lim_{z \to z_1} f(z) = \infty$ then the radius of convergence of the Taylor series is at most $|z_1 - z_0|$, since if it was larger than $|z_1 - z_0|$ the Taylor series would converge to a bounded function on $\mathrm{D}(z_0, |z_1 - z_0|)$, which f is not.

proof after examples....



Examples (We might not have time for all of them on Friday.)

1) Find the Taylor series for $f(z) = e^{z^2}$ at $z_0 = 0$, and its radius of convergence.

$e^{(z^2)}$

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} \quad \forall w$$

$$\Rightarrow e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} \quad \forall z$$

Theorem A power series are unique.
So this is the ans.
$R = \infty$

2) Find the Taylor series for $f(z) = \dfrac{1}{(z-1)^2}$ at $z_0 = 0$, along with its radius of convergence.

$$(1-z)^{-1} = -(1-z)^{-2}(-1) \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad R = 1.$$

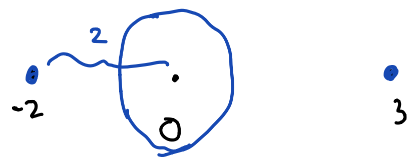
$$\frac{d}{dz} \quad \Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \qquad R = 1$$

Power series Theorem
radius of conv. doesn't change under differentiation & integration.

3) Find the Taylor series for $f(z) = \log(1 + z)$ at $z_0 = 0$, along with its radius of convergence.

4) Find the Taylor series of $f(z) = \dfrac{1}{z^2 - z - 6} = \dfrac{1}{5}\left(\dfrac{1}{z-3} - \dfrac{1}{z+2}\right)$ at $z_0 = 0$, along with its radius of convergence.

Theorem B



$R = 2.$

5) Define $\log(z) = \ln|z| + i\,arg(z)$ on the branch domain $0 < arg(z) < 2\pi$. Find the Taylor series for $\log(z)$ at $z_0 = 1 + i$, and find the radius of convergence using the ratio test for absolute convergence. Explain why your answer may seem surprising at first.

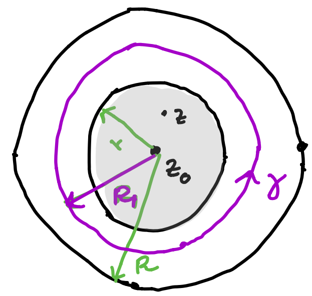
Theorem B If f is analytic in $D(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $D(z_0; R)$. Thus the radius of convergence of the power series is at least R.
Conversely, if $\exists\, z_1$ such that $\lim_{z \to z_1} f(z) = \infty$ then the radius of convergence is at most $|z_1 - z_0|$, since if it was larger the Taylor series would converge to a bounded function near z_1.

proof: Let $|z - z_0| \le r < R_1 < R$, $\gamma(t) = z_0 + R_1 e^{it}$, $0 \le t \le 2\pi$, the circle $|\zeta - z_0| = R_1$.
Then the Cauchy integral formula reads

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$



We use geometric series magic:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \frac{(z - z_0)}{(\zeta - z_0)}}\, d\zeta$$

using the geometric series for $\frac{1}{1 - w}$ with $|w| \le \frac{r}{R_1}$:

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta$$

$$= \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Because $|f|$ is bounded on γ and

$$\frac{|z - z_0|^n}{|\zeta - z_0|^{n+1}} \le \frac{1}{R_1} \left(\frac{r}{R_1} \right)^n,$$

the series which is the integrand converges uniformly on γ so we may interchange the summation with the integration, (and then pull each $(z - z_0)^n$ through the integral:

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \; \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by the Cauchy integral formula for derivatives!

Q.E.D.