so seq. is Cauchy.

Def The series $\sum_{j=1}^{\infty} a_j$ converges *absolutely* if and only if $\sum_{j=1}^{\infty} |a_j| < \infty$. $= L < \infty$. so $\forall \varepsilon > 0 \; \exists N$ s.t. $m < n$, $m, n \geq N$ $\Rightarrow \sum_{j=m+1}^{n} |a_j| < \varepsilon$

Theorem: Absolute convergence implies convergence .

proof:

we'll check orig. seq. is Cauchy.

Let $\varepsilon > 0$. $\exists N$ s.t. $m<n$, $m,n \geq N \overset{?}{\Rightarrow} |S_n - S_m| < \varepsilon$ ↑ use that N

$|S_n - S_m| = \left|\sum_{j=m+1}^{n} a_j\right| \leq \sum_{j=m+1}^{n} |a_j| < \varepsilon$

There is a useful test for uniform convergence of a series of functions on a domain A - namely a uniform absolute convergence test. It's called the *Weierstrass M test* (maybe M is chosen because of the word M*odulus*):

(g_n cont, analytic)

Theorem C (Weierstrass M test) Let $\{g_n(z)\}$, $g_n : A \to \mathbb{C}$. If

$$\forall\, n \in \mathbb{N} \;\exists\, M_n \in \mathbb{R} \text{ such that}$$

$$|g_n(z)| \leq M_n \;\forall\, z \in A$$

and if

$$\sum_{n=1}^{\infty} M_n < \infty$$

then $\sum_{j=1}^{\infty} g_j(z)$ converges uniformly on A. (And in this case, if each g_n is analytic, so is

$g(z) = \sum_{j=1}^{\infty} g_j(z)$.)

proof:

Need to show uniform Cauchy for partial sums.

Let $\varepsilon > 0$. Pick N s.t. $m,n \geq N$, $m<n$

$\Rightarrow \sum_{j=m+1}^{n} M_j < \varepsilon$.

then $|S_n(z) - S_m(z)| = \left|\sum_{j=m+1}^{n} g_j(z)\right| \leq \sum |\;| \leq \sum_{j=m+1}^{n} M_j < \varepsilon$.

$(\forall z \in A)$

∎

Examples

(1) Last week we discussed the Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which converges uniformly absolutely for $\text{Re}(s) > 1 + \varepsilon$, for each positive ε, so is analytic for $\text{Re}(s) > 1$.

(2) ~~Show that~~ Consider

$$\sum_{n=0}^{\infty} z^n$$

Use the Weierstrass M test to show this series converges uniformly on $D(0, r)$ for any $r < 1$, so converges to an analytic function in all of $D(0; 1)$. What analytic function is this? (By the way, this is the most important power series in Complex Analysis. :-))

(3) Show that the series $\sum_{n=0}^{\infty} z^n$ diverges for all $|z| \geq 1$.

(4) Show that the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \dots.$$

converges uniformly for $|z| \leq R$, so converges to an analytic function $\forall z$. Then use the term by term differentiation theorem to show that $f'(z) = f(z)$ and use this to identify $f(z)$.

Math 4200
Friday October 25
3.2 Power series and Taylor series for analytic functions. We'll begin with the examples of power series at the end of Wednesday's notes

Announcements:

Power series

Consider the *power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad\qquad a_n, z_0 \in \mathbb{C}.$$

Theorem:
(1) There exists unique $R \in [0, \infty]$ such that the power series above converges $\forall\, z$ with $|z - z_0| < R$ and diverges for all z with $|z - z_0| > R$. This value of R is called the *radius of convergence* of the power series.

(2) For $r < R$, the convergence of the power series is uniform $\forall\, z \in \mathrm{D}(z_0, r)$. Thus f is analytic in $\mathrm{D}(z_0\,; R)$.

(3) Furthermore, $f'(z)$ can be computed via term by term differentiation; and the antiderivatives $F(z)$ of $f(z)$ can be computed by term by term antidifferentiation. The power series for f' and F have the same radius of convergence R as does the power series f.

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad\qquad \forall\, z \in \mathrm{D}(z_0, R)$$

$$F(z) = F(z_0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} \qquad \forall\, z \in \mathrm{D}(z_0, R)$$

proof: If the radius of convergence exists, it must be unique. Define the potentially different number $R := sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n|\, r^n < \infty\}$. We'll show that this number R satisfies the two required convergence conditions to be the radius of convergence, so it *is* the radius of convergence.

(i) Let $r < R$ and apply the Weirestrass M test in $\overline{\mathrm{D}}(z_0, r)$ to deduce uniform convergence of the power series for $f(z)$ in $\mathrm{D}(z_0, r)$. Thus the power series converges in $\mathrm{D}(z_0, R)$ to an analytic function and we have shown (2), and half of (1).

(ii) Let $|z_1 - z_0| = R_1 > R$. If $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges then the sequence of partial sums is Cauchy, so the n^{th} terms of the series, $a_n (z_1 - z_0)^{n+1} = S_{n+1} - S_n$, must converge to zero, so there is a uniform bound C on the modulus of all the terms $a_n (z_1 - z_0)^n$ in the series. Use this to show that

$$\sum_{n=1}^{\infty} |a_n| r^n < \infty \qquad \forall\, r < R_1$$

which would contradict the *sup* definition of R. Thus the power series for $f(z)$ diverges whenever

$|z - z_0| > R$, and have finished showing (1).

(3) It's possible $R = 0$ in which case the power series only converges at $z = z_0$. Otherwise $R > 0$, and since the power series for $f(z)$ converges in $D(z_0, R)$, and uniformly for any closed subdisk (concentric or not, since each closed subdisk is contained in a closed concentric sub-disk), we deduce from Theorem B′ Wednesday that the term-by-term differentiated power series for $f'(z)$ also converges in $D(z_0, R)$, and uniformly for any closed subdisk. Thus the radius of convergence for the $f'(z)$ series is at least the radius of convergence for the f series. But using the alternate characterization of radius of convergence from part (i), the radius convergence for the series for $f'(z)$ is at most R, since the moduli of the terms in the f' series are larger than in the f series:

$$sup\{r \geq 0 \mid \sum_{n=1}^{\infty} n|a_n| r^n < \infty \} \leq sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n| r^n < \infty \} = R.$$

Thus the radii of convergence for f, f' must be the same. Similarly, the radius of convergence for F must be at least $R > 0$ because $\frac{|a_n|}{n+1} \leq |a_n|$. But the radius of convergence for F and $F' = f$ are equal by our earlier reasoning, so the radius of convergence of F is also R.

QED.

Theorem A. (Uniqueness of power series representations) If f is given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C} \qquad a_n, z_0 \in \mathbb{C}$$

with positive radius of convergence R then

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n = 0, 1, 2, ...$$

so the power series for f centered at z_0 *is* the Taylor series for f centered at z_0. In particular, a given analytic function whose domain of analyticity includes z_0 can have only one power series representation centered at z_0.

proof: We know from the previous Theorem that we have

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad \forall\, z \in \mathrm{D}(z_0, R)$$

and inductively, for $k \in \mathbb{N}$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n (z - z_0)^{n-k} \qquad \forall\, z \in \mathrm{D}(z_0, R),\ \forall\, k \in \mathbb{N}.$$

evaluating at z_0 only the first term in the series is nonzero, so

$$f^{(k)}(z_0) = k!\, a_k \quad \Rightarrow a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Theorem B If f is analytic in $\mathrm{D}(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $\mathrm{D}(z_0; R)$. Thus the radius of convergence of the Taylor series is at least R. Thus if $\exists\, z_1$ such that $\lim_{z \to z_1} f(z) = \infty$ then the radius of convergence of the Taylor series is at most $|z_1 - z_0|$, since if it was larger than $|z_1 - z_0|$ the Taylor series would converge to a bounded function on $\mathrm{D}(z_0, |z_1 - z_0|)$, which f is not.

proof after examples....

Examples (We might not have time for all of them on Friday.)

1) Find the Taylor series for $f(z) = e^{z^2}$ at $z_0 = 0$, and its radius of convergence.

2) Find the Taylor series for $f(z) = \dfrac{1}{(z-1)^2}$ at $z_0 = 0$, along with its radius of convergence.

3) Find the Taylor series for $f(z) = \log(1 + z)$ at $z_0 = 0$, along with its radius of convergence.

4) Find the Taylor series of $f(z) = \frac{1}{z^2 - z - 6} = \frac{1}{5}\left(\frac{1}{z-3} - \frac{1}{z+2}\right)$ at $z_0 = 0$, along with its radius of convergence.

5) Define $\log(z) = \ln|z| + i\,arg(z)$ on the branch domain $0 < arg(z) < 2\pi$. Find the Taylor series for $\log(z)$ at $z_0 = 1 + i$, and find the radius of convergence using the ratio test for absolute convergence. Explain why your answer may seem surprising at first.

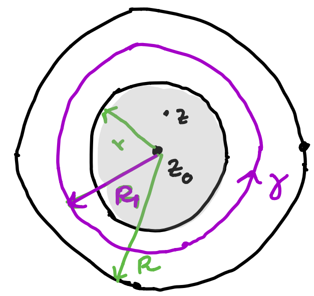
Theorem B If f is analytic in $\mathrm{D}(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $\mathrm{D}(z_0; R)$. Thus the radius of convergence of the power series is at least R. Conversely, if $\exists\, z_1$ such that $\lim_{z \to z_1} f(z) = \infty$ then the radius of convergence is at most $|z_1 - z_0|$, since if it was larger the Taylor series would converge to a bounded function near z_1.

proof: Let $|z - z_0| \leq r < R_1 < R$, $\gamma(t) = z_0 + R_1 e^{i\,t}$, $0 \leq t \leq 2\,\pi$, the circle $\zeta - z_0 = R_1$. Then the Cauchy integral formula reads

$$f(z) = \frac{1}{2\,\pi\,i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$



We use geometric series magic:

$$f(z) = \frac{1}{2\,\pi\,i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\,\pi\,i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \frac{(z - z_0)}{(\zeta - z_0)}}\, d\zeta$$

using the geometric series for $\frac{1}{1 - w}$ with $|w| \leq \frac{r}{R_1}$:

$$= \frac{1}{2\,\pi\,i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta$$

$$= \frac{1}{2\,\pi\,i} \int_{\gamma} \sum_{n=0}^{\infty} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Because $|f|$ is bounded on γ and

$$\frac{|z - z_0|^n}{|\zeta - z_0|^{n+1}} \leq \frac{1}{R_1} \left(\frac{r}{R_1} \right)^n,$$

the series which is the integrand converges uniformly on γ so we may interchange the summation with the integration, (and then pull each $(z - z_0)^n$ through the integral:

$$f(z) = \frac{1}{2\,\pi\,i} \sum_{n=0}^{\infty} (z - z_0)^n \int_{\gamma} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \; \frac{1}{2\,\pi\,i} \int_{\gamma} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by the Cauchy integral formula for derivatives!

Q.E.D.