Math 4200
Wednesday October 23
3.1 Infinite sum expressions for analytic functions and their derivatives; introduction to power series.

Announcements: We didn't get to the Poisson integral formula for harmonic functions in the disk on Monday. We'll postpone or cancel that discussion so that we can begin Chapter 3.

- last page is HW
- several pages before that is a list of key ideas from 2.3-2.5 (that we won't go over in class).

Chapter 3: Series representations for analytic functions. Section 3.1: Sequences and series of analytic functions.

Recall a key analysis theorem which we proved and used in our discussion of uniform limits of analytic functions last week:

Theorem Let $A \subseteq \mathbb{C}, f_n : A \to \mathbb{C}$ continuous, $n = 1, 2, 3 \ldots$ If $\{f_n\} \to f$ uniformly, then f is continuous.
(The same proof would've worked for $A \subseteq \mathbb{R}^k$, $F_n : A \to \mathbb{R}^p$, $\{F_n\} \to F$ uniformly.)

Crux $z_0 \in A$. $|z - z_0|$ small

$$|f(z) - f(z_0)| \le \underbrace{|f(z) - f_N(z)|}_{1} + \underbrace{|f_N(z) - f_N(z_0)|}_{2} + \underbrace{|f_N(z_0) - f(z_0)|}_{3}$$

N large

Let $\varepsilon > 0$ Pick N s.t. 1, 3 $< \varepsilon/3$ Uniform conv.
then pick $\delta > 0$ s.t. 2 $< \varepsilon/3$ for $|z - z_0| < \delta$ cont. of f_N.

Corollary Let $A \subseteq \mathbb{C}, f_n : A \to \mathbb{C}$ continuous, $n = 1, 2, 3 \ldots$ If $\{f_n\}$ is *uniformly Cauchy*, then there exist a continous limit function $f: A \to \mathbb{C}$, with $\{f_n\} \to f$ uniformly.

$\forall \varepsilon > 0 \ \exists N$ s.t. $n, m \ge N$
$|f_n(z) - f_m(z)| < \varepsilon \ \forall z \in A$.

pf: so $\forall z \in A$, $\{f_n(z)\}$ is Cauchy
- $\mathbb{C}$ complete $\Rightarrow \{f_n(z)\} \to L := f(z)$.

- $\{f_n\} \to f$ uniformly.
  Let $\varepsilon > 0$. Let N as in def. of unif. Cauchy.
  $\forall n, m \ge N$, $|f_n(z) - f_m(z)| < \varepsilon$
  Let $m \to \infty \Rightarrow |f_n(z) - f(z)| \le \varepsilon$. $\forall n \ge N$, $\forall z \in A$.

Theorem A (essentially from last week, via Morera's Theorem) Let $A \subseteq \mathbb{C}$ open, $f_n : A \to \mathbb{C}$ analytic, and $\{f_n\}$ *uniformly Cauchy*. Then $\exists f$ with $\{f_n\} \to f$ uniformly, and f is analytic.

pf Rectangle lemma holds for each f_n, each $R \subset A$, (closed rect)
i.e. $\displaystyle\oint_{\partial R} f_n(z)\,dz = 0$ since f_n's are analytic.
$\{f_n\} \to f$ uniformly $\Rightarrow \displaystyle\oint_{\partial R} f(z)\,dz = 0$
Morera $\Rightarrow f$ is analytic.

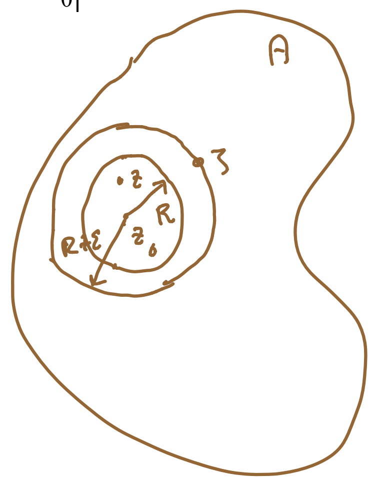
Theorem B (includes statement about convergence of the derivatives) Let $A \subseteq \mathbb{C}$ open, $f_n : A \to \mathbb{C}$ analytic; $\{f_n(z)\} \to f(z)\ \forall\, z \in A$; $\{f_n\} \to f$ uniformly on each closed disk $\bar{D}(z_0\,;R) \subseteq A$. Then

(1) f is analytic on A

(2) Furthermore, the derivatives $f_n{}'(z) \to f'(z)$ and the convergence is uniform on each closed disk $\bar{D}(z_0\,;R) \subseteq A$.

proof: (1) follows from Theorem A, applied to each subdisk $D(z_0\,;R)$ with $\bar{D}(z_0\,;R) \subseteq A$.

For (2), Let $\bar{D}(z_0\,;R) \subseteq A$, pick $\varepsilon > 0$ so that also $\bar{D}(z_0\,;R+\varepsilon) \subseteq A$. Then for $|z - z_0| \le R$ use the Cauchy integral formulas for derivatives on the circle of radius $R + \varepsilon$ and compare:



C.I.F. $$f_n{}'(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R+\varepsilon} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R+\varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$|f_n{}'(z)\; f'(z)| = \frac{1}{2\pi} \left| \int_{|\zeta - z_0| = R+\varepsilon} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\, d\zeta \right|$$

$$\le \cancel{\frac{1}{2\pi}} \max\left\{ \underbrace{\left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \right|}_{} \text{ s.t. } |\zeta - z_0| = R+\varepsilon \right\} \cancel{2\pi} (R+\varepsilon)$$

$$\le \frac{\max\{ |f_n(\zeta) - f(\zeta)| \text{ s.t. } |\zeta - z_0| = R+\varepsilon \}}{\min\{ |\zeta - z|^2 \text{ s.t. } |\zeta - z_0| = R+\varepsilon \}} (R+\varepsilon)$$

$$\le \frac{\max\{ |f_n(\zeta) - f(\zeta)| \text{ s.t. } |\zeta - z_0| = R+\varepsilon \}}{\varepsilon^2 \leftarrow \text{fixed}} (R+\varepsilon) \qquad |\zeta - z| \ge |\zeta| - |z| \ge \varepsilon.$$

$\longrightarrow 0$ indep. of $z \in \overline{D(0;R)}$. ∎

Recall from analysis the following correspondence between sequences $\{f_n(z)\}$ and series $\sum_{j=1}^{\infty} g_j(z)$:

- Each series $\sum_{j=1}^{\infty} g_j(z)$ corresponds a sequence of partial sums $\{S_n(z)\}$, with $S_n(z) := \sum_{j=1}^{n} g_j(z)$.
- Each sequence $\{f_n(z)\}$ can be rewritten as an infinite series $\sum_{j=1}^{\infty} g_j(z)$ with partial sums $S_n = f_n$ if

we define

$$\begin{aligned} g_1(z) &:= f_1(z) \\ g_2(z) &= f_2(z) - f_1(z) \\ &\vdots \\ g_n(z) &= f_n(z) - f_{n-1}(z). \end{aligned}$$

<u>Def</u> The series $\sum_{j=1}^{\infty} g_j(z)$ *converges uniformly* on the domain A (alternately *at a point* $z \in A$) if and only if the sequence of partial sums $S_n(z) = \sum_{j=1}^{n} g_j(z)$ converges uniformly on A (alternately at a point $z \in A$).

<u>Theorem B′</u> (Theorem B restated for series): Let $A \subseteq \mathbb{C}$ open, $g_n : A \to \mathbb{C}$ analytic; $S_n(z) = \sum_{j=1}^{n} g_j(z) \to f(z) = \sum_{j=1}^{\infty} g_j(z)$ $\forall\, z \in A$; $\{S_n\} \to f$ uniformly on each closed disk $\bar{\mathrm{D}}(z_0; R) \subseteq A$. Then

(1) $f(z) = \sum_{j=1}^{\infty} g_j(z)$ is analytic on A since $\{S_n(z)\} \to f(z)$ uniformly on each $\overline{D(z_0;R)} \subset A$.

(2) $\frac{d}{dz} f(z) = \frac{d}{dz} \sum_{j=1}^{\infty} g_j(z) = \sum_{j=1}^{\infty} g_j{}'(z)$. Furthermore, the convergence of $\sum_{j=1}^{\infty} g_j{}'(z)$ to $f'(z)$ is uniform on each closed disk $\bar{\mathrm{D}}(z_0; R) \subseteq A$. (In other words, we can differentiate the series term by term.)

$\{S_n(z)\} \to f(z)$ uniformly on each $\overline{D(z_0;R)} \subset A$

$\{S_n'(z)\} \to f'(z)$ " " " " "

from Thm B.

$S_n(z) = \sum_{j=1}^{n} g_j(z)$

$S_n'(z) = \sum_{j=1}^{n} g_j'(z)$ ∎

so seq. is Cauchy.

Def The series $\sum_{j=1}^{\infty} a_j$ converges *absolutely* if and only if $\sum_{j=1}^{\infty} |a_j| < \infty$. $= L < \infty$, so $\forall \varepsilon > 0 \ \exists N$ s.t. $m < n$, $m, n \geq N$ $\Rightarrow \sum_{j=m+1}^{n} |a_j| < \varepsilon$

Theorem: Absolute convergence implies convergence .

proof:

we'll check orig. seq. is Cauchy. use that N

Let $\varepsilon > 0$. $\exists N$ s.t. $m < n$, $m, n \geq N \overset{?}{\Rightarrow} \underbrace{|S_n - S_m|}_{\left|\sum_{j=m+1}^{n} a_j\right| \leq \sum_{j=m+1}^{n} |a_j| < \varepsilon} < \varepsilon$

There is a useful test for uniform convergence of a series of functions on a domain A - namely a uniform absolute convergence test. It's called the *Weierstrass M test* (maybe M is chosen because of the word M*odulus*):

(g_n cont, analytic)

Theorem C (Weierstrass M test) Let $\{g_n(z)\}$, $g_n : A \to \mathbb{C}$. If

$$\forall\ n \in \mathbb{N}\ \exists\ M_n \in \mathbb{R} \text{ such that}$$

$$|g_n(z)| \leq M_n\ \forall\ z \in A$$

and if

$$\sum_{n=1}^{\infty} M_n < \infty$$

then $\sum_{j=1}^{\infty} g_j(z)$ converges uniformly on A. (And in this case, if each g_n is analytic, so is

$g(z) = \sum_{j=1}^{\infty} g_j(z)$.)

proof:

Need to show uniform Cauchy for partial sums.

Let $\varepsilon > 0$. Pick N s.t. $m, n \geq N$, $m < n$

$\Rightarrow \sum_{j=m+1}^{n} M_j < \varepsilon$.

then $|S_n(z) - S_m(z)| = \left|\sum_{j=m+1}^{n} g_j(z)\right| \leq \sum | \ | \leq \sum_{j=m+1}^{n} M_j < \varepsilon$.

$(\forall z \in A)$

∎

Examples

(1) Last week we discussed the Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which converges uniformly absolutely for $\mathrm{Re}(s) > 1 + \varepsilon$, for each positive ε, so is analytic for $\mathrm{Re}(s) > 1$.

(2) Consider ~~Show that~~

$$\sum_{n=0}^{\infty} z^n$$

Use the Weierstrass M test to show this series converges uniformly on $D(0, r)$ for any $r < 1$, so converges to an analytic function in all of $D(0; 1)$. What analytic function is this? (By the way, this is the most important power series in Complex Analysis. :-))

(3) Show that the series $\sum_{n=0}^{\infty} z^n$ diverges for all $|z| \geq 1$.

(4) Show that the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots.$$

converges uniformly for $|z| \leq R$, so converges to an analytic function $\forall\, z$. Then use the term by term differentiation theorem to show that $f'(z) = f(z)$ and use this to identify $f(z)$.

Appendix: Key results of Chapter 2.3-2.5. We'll be using many of these in Chapter 3.

Cauchy's Theorem (Deformation theorem version, section 2.5). Let $f: A \to \mathbb{C}$ analytic.
a) Let $\gamma: [a, b] \to A$ a (piecewise C^1) closed contour. If γ is homotopic to a point in A as closed curves, then

$$\int_\gamma f(z)\, dz = 0.$$

b) If γ_1, γ_2 are homotopic with fixed endpoints in A, then

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

(proofs used the homotopy lemma, which made use of the local antidifferentiation theorem, which used Goursat's rectangle lemma.)

Index The signed number of times a closed contour γ winds around z_0 can be counted with the index formula

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\, dz$$

(We showed that for any continuous closed contour not containing z_0 there's a unique way to measure how the polar angle from z_0 to $\gamma(t)$ changes as one traverses γ, and dividing that total change by 2π is the definition of the index. We showed that for a piecewise C^1 contour, the contour integral expression above computes the same integer.)

Cauchy Integral Formula Let $f: A \to \mathbb{C}$ analytic. Let γ be a closed contour homotopic to a point in A. Then for $z \notin \gamma$,

$$f(z)\, I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

(We applied the deformation theorem and local antiderivative theorem with the modified rectangle lemma, forthe auxillary function

$$G(\zeta) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$)

Cauchy Integral Formula for derivatives Let $f: A \to \mathbb{C}$ analytic. Then f is infinitely differentiable. And, for γ be a closed contour homotopic to a point in A and $z \notin \gamma$,

$$f'(z)\, I(\gamma; z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f^{(n)}(z)\, I(\gamma; z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

(We used the fact that if integrands of contour integrals are converging uniformly, then so are the contour integrals.)

Estimates: In case γ is the index one circle of radius R centered at z,

$$|f^{(n)}(z)| \le \frac{n!}{R^n} \max \{|f(\zeta)| \text{ s.t. } |\zeta - z| = R\}.$$

Corollaries Liouville's Theorem: Bounded entire functions are constant.

Theorem Entire functions with moduli that that are bounded by $C|z|^n$ with $n \in \mathbb{N}$, for $|z|$ large, are polynomials of degree at most n.

Fundamental Theorem of Algebra: Every polynomial $p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ factors as

$$p(z) = (z - \alpha_1)(z - \alpha_2)\ldots.(z - \alpha_n)$$

Morera's Theorem Let $f: A - \mathbb{C}$ continuous and supposed the rectangle lemma holds for every closed rectangle $R \subseteq A$,

$$\int_{\delta A} f(z)\, dz = 0.$$

Then f is analytic in A.

(The rectangle lemma means f has local antiderivatives F, but these F are twice complex differentiable, so f is complex differentiable.)

Key corollary for Chapter 3: Let $\{f_n\}, f_n : A \to \mathbb{C}$ analytic, $\{f_n\} \to f$ uniformly on A. Then f is analytic on A.

(We proved that uniform limits of continuous functions are continuous, so that we can compute contour integrals for the limit function f. And since each f_n is analytic, and $\{f_n\} \to f$ uniformly, the rectangle lemma hypothesis of Morera's Theorem is satisfied by f, so f is analytic.)

Mean value properties

Let $f: A \to \mathbb{C}$ analytic, $\bar{D}(z_0; R) \subseteq A$. Then the value of f at z_0 is the average of the values of f on the concentric circle of radius R about z_0:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + R\,e^{i\theta}\right) d\theta$$

Let $u : A \to \mathbb{R}$ harmonic and C^2, $\bar{D}(z_0; R) \subseteq A$. Then the value of u at (x_0, y_0) is the average of the values of u on the concentric circle of radius R about z_0:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(x_0 + R\cos(\theta), y_0 + R\sin(\theta)\right) d\theta$$

Theorem (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f: A \to \mathbb{C}$ be analytic, $f: \bar{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \bar{A}}\{|f(z)|\,| = \max_{z \in \delta A}\{|f(z)|\} := M.$$

Furthermore if $\exists\ z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function on A.

Theorem (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u : A \to \mathbb{R}$ be harmonic and C^2, $u : \bar{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x,y) \in \bar{A}}\{u(x,y)\} = \max_{(x,y) \in \delta A}\{u(x,y)\} := M,$$
$$\min_{(x,y) \in \bar{A}}\{u(x,y)\} = \min_{(x,y) \in \delta A}\{u(x,y)\} := m,$$

Furthermore if $\exists\ (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then u is a constant function on A.

Theorem Let $f : \overline{\mathrm{D}}(0; 1) \to \overline{\mathrm{D}}(0; 1)$ be a conformal diffeomorphism of the closed unit disk (i.e. f and f^{-1} are each conformal). Then recording

$$f(0) = z_0,$$

f must be a composition

$$f(z) = g_{z_0}\left(e^{i\theta} z\right)$$

for some choice of θ and the specific Mobius transformations $g_{z_0}(z)$

$$g_{z_0}(z) = \frac{z_0 + z}{1 + \overline{z_0}\, z}.$$

Theorem (Poisson integral formula for the unit disk) Let $u \in C^2(\mathrm{D}(0; 1)) \cap C\left(\overline{\mathrm{D}}(0; 1)\right)$, and let u be harmonic in $\mathrm{D}(0; 1)$. Then the Poisson integral formula recovers the values of u inside the disk, from the boundary values. It may be expressed equivalently in complex form or real form. For $z_0 = x_0 + i y_0 = r e^{i\varphi}$ with $|z_0| < 1$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z_0|^2}{\left|z_0 - e^{i\theta}\right|^2} u\left(e^{i\theta}\right) d\theta$$

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi} \int_0^{\pi} \frac{1 - r^2}{r^2 - 2r\cos(\theta - \varphi) + 1} u(\cos(\theta), \sin(\theta))\, d\theta$$

Math 4200-001
Week 9 concepts and homework
3.1-3.2
Due Wednesday October 30 at start of class.

3.1 4, 6, 7, 12, 13, 14
3.2 2, 3, 4, 5c, 7, 13, 14, 18, 19, 20