Math 4200
Monday October 21
2.5 maximum and minimum principles for harmonic functions; conformal diffeomorphisms of the disk via the maximum modulus principle, with application to the Poisson integral formula for harmonic functions.

Announcements: HW questions...

- 2.5.18 f entire, $\mathrm{Im}(f) \le 0 \Rightarrow f$ constant.

  Liouville: g entire & bounded $\Rightarrow g$ const.
  Use f & compose it with another fcn, to get a g for Liouville.
- some other HW problems (also) connect to today's class.
- Start Chapter 3 on Wed $\longleftrightarrow$ relating analytic fcns to power series.

On Friday we proved the maximum modulus principle for analytic functions. The key tool was the mean value property.

<u>Theorem</u> (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f : A \to \mathbb{C}$ be analytic, $f : \overline{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \overline{A}}\{|f(z)|\} = \max_{z \in \delta A}\{|f(z)|\} := M,$$

i.e. the maximum modulus of $f(z)$ occurs on the boundary of A. (Recall that for an open set A, the boundary $\delta A = \overline{A} \setminus A$. For general sets the boundary is the collection of points in the closure of the set as well as in the closure of its complement.)

Furthermore if $\exists\ z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function on A.

One can use the mean value property for harmonic functions to prove the maximum and minimum modulus theorems for harmonic functions:

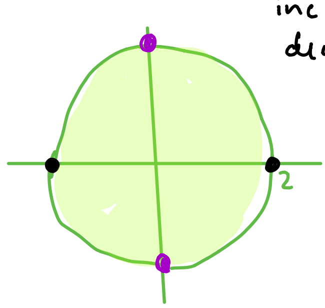
<u>Theorem</u> (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u : A \to \mathbb{R}$ be harmonic and C^2, $u : \overline{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x,y) \in \overline{A}}\{u(x,y)\} = \max_{(x,y) \in \delta A}\{u(x,y)\} := M,$$
$$\min_{(x,y) \in \overline{A}}\{u(x,y)\} = \min_{(x,y) \in \delta A}\{u(x,y)\} := m,$$

Furthermore if $\exists\ (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then u is a constant function on A.

<u>Example</u>: $u(x,y) = x^2 - y^2$ is harmonic. ✓ Where are the maximum and minimum values of u attained, on $D(0;2)$?

inc in $|x|$
dec in $|y|$



$u(2,0) = u(-2,0) = 4$ = Max value.

$u(0,\pm 2) = -4$ = min value.

proof: The maximum principle implies the minimum principle, since the minimum principle for $u(x, y)$ is equivalent to the maximum principle for $v(x, y) = -u(x, y)$. In other words, minimum values for $u(x, y)$ correspond to maximum values for $-u(x, y)$, and u is harmonic if and only if $-u$ is. So we'll focus on the maximum principle. The key tool is the mean value principle for harmonic functions: For every closed disk in A, the average value of u on the bounding circle equals the value at the center. Can you see how the proof goes, if we follow the outline of the maximum modulus principle proof?

The maximum value M must occur either in the open domain A or on the boundary. The Theorem follows if we show that whenever there is an interior point $(x_0, y_0) \in A$ with $u(x_0, y_0) = M$, then actually $u(x, y) = M \ \forall \ (x, y) \in A$.

Let $B = \{ (x, y) \in A \mid u(x, y) = M \}$.

u is continuous, $\Rightarrow u^{-1}\{M\}$ is relatively closed in A.

Because u is continuous, B is closed in A. If we can show B is open, then B is either the empty set or all of A since A is connected. So, suppose B is not empty and let $z_0 \in B$.

Method 1: Let $D(z_0; \rho) \subseteq A$ and show $D(z_0; \rho) \subseteq B$ by using the mean value property for each $0 < r < \rho$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta$$

$$M = u(x_0, y_0) \le \frac{1}{2\pi} \int_0^{2\pi} M \, d\theta \le \frac{1}{2\pi} 2\pi M = M$$

if $u(x_0 + r\cos\theta, y_0 + r\sin\theta) < M$ for some θ_0
then it's strictly $< M$ on some interval $(\theta_0 - \delta, \theta_0 + \delta)$
so we would get a strict inequality
$M < M$ $\times$. Thus $u \equiv M$ on circle of radius r $\forall\, 0 < r < \rho$ $\Rightarrow u \equiv M$ on $D(z_0; \rho)$.

Method 2: Let $\bar{D}(z_0; \rho) \subseteq A$ and show $D(z_0; \rho) \subseteq B$ via slick shortcut in text: Let $v(x, y)$ be the harmonic conjugate of u in $\bar{D}(z_0; \rho)$, and write $f(z) = u + i\, v$ for the corresponding analytic function. Then apply the maximum modulus principle to the composition $g(z) = e^{f(z)}$, on $\bar{D}(z_0; \rho)$.

by max mod. principle.

$$\max\{|g(z)| \text{ s.t. } z \in \bar{D}[z_0; \rho]\} = \max\{|g(z)| \text{ s.t. } |z - z_0| = \rho\}$$

$$\| $$

$$|g(z)| = |e^{u+iv}| = e^u$$

max is $e^M = e^{u(x_0, y_0)}$

$|g(z)|$ attains its max inside disk
$\Rightarrow |g(z)|$ is const on disk.
$\Rightarrow e^u$ is const $\Rightarrow u \equiv M$.

Application of the maximum modulus principle, related to section 5.2, which also yields a different proof of the *Poisson integral formula* for harmonic functions than the text's, in the current section 2.5

Question. Consider $D = D(0;1) \subseteq \mathbb{C}$. What are all possible invertible conformal transformations $f : \bar{D} \to \bar{D}$? In other words, so that f, f^{-1} are each analytic bijections of the closed unit disk, $f : S^1 \to S^1$, $S^1 = \{|z|=1\}$

Step 1 What if we require $f(0) = 0$? Then consider

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases} \qquad \frac{1}{g(z)} = \begin{cases} \dfrac{z}{f(z)} & z \neq 0 \\ \dfrac{1}{f'(0)} & z = 0 \end{cases}$$

Since g is analytic in $\bar{D}$ except at the point $z = 0$ where it is continuous, the modified rectangle lemma and Morera's Theorem prove that g is analytic in the closed disk (i.e. in a slightly larger open disk). The same reasoning applies to $\dfrac{1}{g(z)}$. Use the maximum modulus principle for $g(z)$ and for $\dfrac{1}{g(z)}$ to show that $f(z) = e^{i\theta} z$ are the only conformal diffeomorphisms in this case. Not very many!!!

$$\max\{|g(z)| \text{ s.t. } z \in \overline{D(0;1)}\} = \max\{|g(z)| \text{ s.t. } |z|=1\} = 1.$$

$$|g(z)| = \frac{|f(z)|}{|z|} = 1$$

So $(z \neq 0) \Rightarrow \left|\dfrac{f(z)}{z}\right| \leq 1$

$|f(z)| \leq |z|.$ ∎

$$\max\left\{\frac{1}{|g(z)|} \text{ s.t. } z \in \overline{D(0;1)}\right\} = \max\left\{\frac{|z|}{|f(z)|} \text{ s.t. } |z|=1\right\} = 1.$$

So $(z \neq 0) \Rightarrow \left|\dfrac{z}{f(z)}\right| \leq 1 \Rightarrow |z| \leq |f(z)|.$ ∎

$\Rightarrow \left|\dfrac{f(z)}{z}\right| \equiv 1.$ $(|g(z)| \equiv 1)$ so $g(z) = \dfrac{f(z)}{z} = C = e^{i\theta}$

with modulus 1.

$$\boxed{f(z) = e^{i\theta} z}$$

modulus $\frac{1}{|z_0|} > 1$

Step 2 Consider the Mobius transformation (see p. 340, Chapter 5.2; also a first-week homework problem):

g is analytic except where $1 + \bar{z}_0 z = 0$

$\bar{z}_0 z = -1$; $z = -\frac{1}{\bar{z}_0}$

$$g'(z) = \frac{1 \cdot (1 + \cancel{\bar{z}_0 z}) - (z_0 + \cancel{z})\bar{z}_0}{(1 + \bar{z}_0 z)^2}$$

a) $g(z) = \dfrac{z_0 + z}{1 + \bar{z}_0 z}$, $\quad z_0 \in \mathrm{D}(0; 1)$

g_{z_0}

$g(0) = z_0 \in D(0;1)$.

$$g'(z) = \frac{1 - |z_0|^2}{(1 + \bar{z}_0 z)^2} \neq 0.$$

Show $g(z)$ is conformal in the closed unit disk: $g'(z) = \dfrac{1 - |z_0|^2}{\left(1 + \bar{z}_0 z\right)^2}$ exists and is non zero in the closed unit disk.

Notice that $g(0) = z_0$. Show that g transforms the unit circle to the unit circle, so that by the maximum modulus principle, $|g(z)| < 1 \;\forall\; z \in \mathrm{D}(0; 1)$.

$$|g(z)|^2 = \frac{z_0 + z}{1 + \bar{z}_0 z} \cdot \frac{\bar{z}_0 + \bar{z}}{1 + z_0 \bar{z}} = \frac{|z_0|^2 + z\bar{z}_0 + z_0\bar{z} + |z|^2}{1 + \bar{z}_0 z + z_0 \bar{z} + |z_0|^2 |z|^2} = 1!$$

$|z| = 1$

$|z| = 1$

b) If we denote the Mobius transform g in part (a) by g_{z_0} because of the image of the origin, solve the equation

$$\frac{z_0 + z}{1 + \bar{z}_0 z} = w$$

: $z_0 + z = w(1 + \bar{z}_0 z)$

$z - w\bar{z}_0 z = w - z_0$

for w to see that the inverse function to $g_{z_0}(z)$ is given by the Mobius transformation

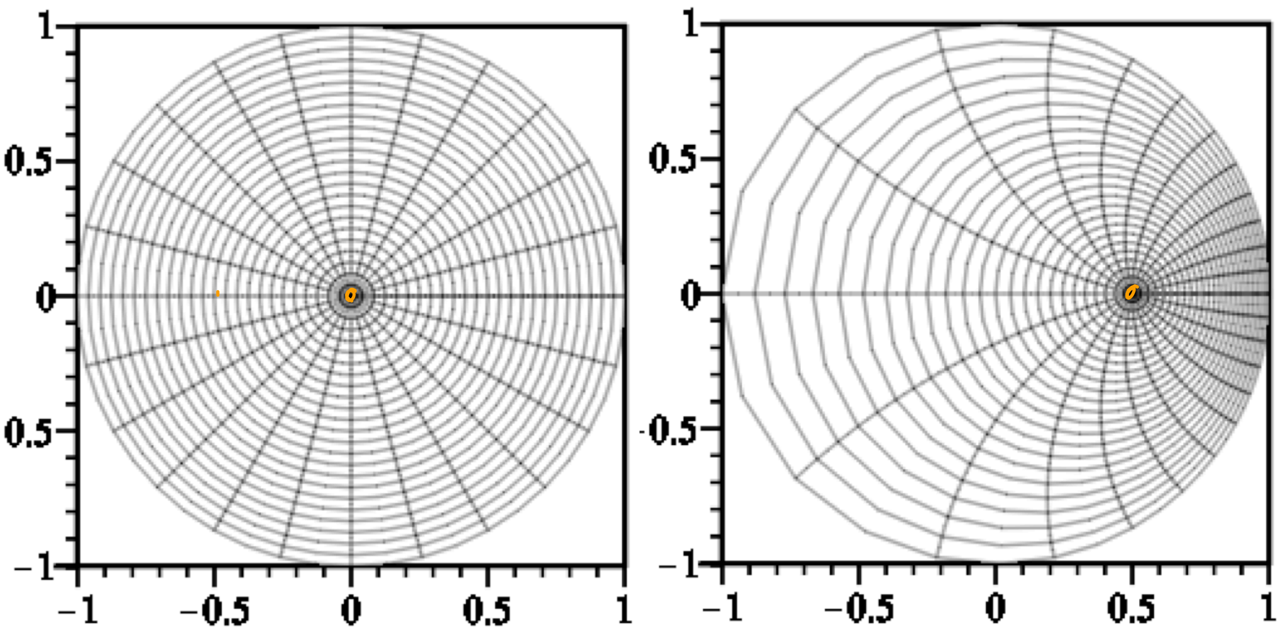
$z[1 - w\bar{z}_0] = -z_0 + w$

$$h(w) = g_{-z_0}(w) = \frac{-z_0 + w}{1 - \bar{z}_0 w}.$$

$$\boxed{z = \frac{-z_0 + w}{1 - \bar{z}_0 w}}$$

Here's a Maple picture of how $g_{.5}(z)$ transforms circles concentric to the origin, and rays through the origin. You'll notice that the images of the circles are circles, and the images of the rays are circles (or rays) that hit the unit circle orthogonally. This is not an accident. It turns out that these *Mobius transformations* g_{z_0} are the *isometries* of the *hyperbolic disk*, in *hypberbolic geometry*. (Another circle of ideas for a potential class project.)

$$g_{.5}(z) = \frac{.5+z}{1+.5z}$$



Step 3: Combine steps 1 and 2, to show that every conformal diffeomorphism of the unit disk with

$$f(0) = z_0$$

can be written as

$$f(z) = g_{z_0}\left(e^{i\theta} z\right)$$

for some choice of θ and the Mobius transformations $g_{z_0}(z)$ with $z_0 \in D(0;1)$, from the previous page,

$$g_{z_0}(z) = \frac{z_0 + z}{1 + \bar{z}_0 z}.$$

Not very many!

$$f(0) = z_0$$
$$g_{z_0}(0) = z_0$$

$$\Rightarrow g_{z_0}^{-1}(f(0)) = g_{z_0}^{-1}(z_0) = 0.$$

so $g_{z_0}^{-1} \circ f(z)$ is conformal diffeo of unit disk.

$= g_{-z_0} \circ f(z)$ $\quad 0 \mapsto 0.$

step 1 $\Rightarrow g_{-z_0} \circ f(z) = e^{i\theta} z$

$$\Rightarrow f(z) = g_{z_0}(e^{i\theta} z)$$

∎

pick up on Wed.

Application to harmonic function theory (in partial differential equations). There is an analog of the Cauchy integral formula for harmonic functions, that expresses the value of a harmonic function inside a domain in terms of an integral over the boundary which uses the harmonic function's boundary values. It's much messier to write down than the Cauchy integral formula in general - if you wanted to take the real part of the Cauchy integral formula you'd also need to know the boundary values of the conjugate to the harmonic function, to deduce the values of the harmonic function in the interior, so you can't just use the CIF, like we did for the mean value property. In the case where the domain is the unit disk (or a scaled disk), this analog to the CIF is known as the *Poisson integral formula*.

Theorem (Poisson integral formula for the unit disk) Let $u \in C^2(D(0;1)) \cap C(\bar{D}(0;1))$, and let u be harmonic in $D(0;1)$. Then the Poisson integral formula recovers the values of u inside the disk, from the boundary values. It may be expressed equivalently in complex form or real form. For $z_0 = x_0 + i\,y_0 = r\,e^{i\varphi}$ with $|z_0| < 1$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z_0|^2}{|z_0 - e^{i\theta}|^2}\, u(e^{i\theta})\,d\theta$$

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi}\int_0^{\pi} \frac{1 - r^2}{r^2 - 2r\cos(\theta - \varphi) + 1}\, u(\cos(\theta), \sin(\theta))\,d\theta$$

* First, check why the CIF formula wouldn't work directly, unless we knew the harmonic conjugate.

** But we do know the mean value property, and we can combine this with the Mobius transformations on the previous page! (Actually we only know the mean value property if u is harmonic on a slightly larger disk than $D(0;1)$, but it also holds for harmonic $u \in C^2(D(0;1)) \cap C(\bar{D}(0;1))$, by a rescaling, limiting process. In any case, consider

$$g_{z_0}(z) = \frac{z_0 + z}{1 + \bar{z}_0\, z}$$

from the previous page. Then $u(g(z))$ is harmonic on the unit disk (do you remember why, from a Chapter 1 homework problem?). So

$$u(z_0) = u\left(g_{z_0}(0)\right) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right)d\alpha$$

by the mean value property. Now we just change variables, and after some computations out pops the Poisson integral formula:

$$g_{z_0}\left(e^{i\alpha}\right) = e^{i\theta}$$

$$g_{-z_0}\left(e^{i\theta}\right) = e^{i\alpha}$$

So,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right) d\alpha$$
$$= \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right)\alpha'(\theta)\, d\theta.$$

To get $\alpha'(\theta)$ we differentiate e.g. the second change of variables formula, using the chain rule for curves and regular Calculus

$$\frac{d}{d\theta}\left(g_{-z_0}\left(e^{i\theta}\right)\right) = \frac{d}{d\theta}\left(e^{i\alpha}\right)$$
$$g_{-z_0}{}'\left(e^{i\theta}\right) i\, e^{i\theta} = i\, e^{i\alpha}\, \alpha'(\theta).$$

From the previous page,

$$g_{-z_0}{}'(z) = \frac{1-|z_0|^2}{\left(1-\bar{z}_0 z\right)^2}$$

so

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0 e^{i\theta}\right)^2}\, i\, e^{i\theta} = i\, \frac{-z_0+e^{i\theta}}{1-\bar{z}_0 e^{i\theta}}\, \alpha'(\theta)$$

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0 e^{i\theta}\right)}\, \frac{e^{i\theta}}{-z_0+e^{i\theta}} = \alpha'(\theta)$$
$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0 e^{i\theta}\right)\left(1-z_0 e^{-i\theta}\right)} = \alpha'(\theta) = \frac{1-|z_0|^2}{\left|z_0-e^{i\theta}\right|^2}$$

QED!!

Harmonic functions exist and are uniquely determined by their boundary values, even if the boundary values are only piecewise continuous....in the disk the harmonic functions can be expressed using Fourier series, or with the Poisson integral formula, and describe various physical phenomena, such as equilibrium temperature distributions in 2-dimensional plates having controlled boundary temperatures....also related to random walk phenomena in probability, other applications.

http://mathfaculty.fullerton.edu/mathews/c2003/DirichletProblemDiskMod.html

Extra Example 1. Find the function $u(x, y) = u(r\cos\theta, r\sin\theta)$ that is harmonic in the unit disk $D_1(0) = \{z : |z| < 1\}$,

and takes on the boundary values $u(\cos\theta, \sin\theta) = U(\theta) = \begin{cases} 1, & \text{for } \frac{\pi}{2} < t < \pi, \\ 0, & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ -1, & \text{for } -\pi < t < -\frac{\pi}{2}. \end{cases}$

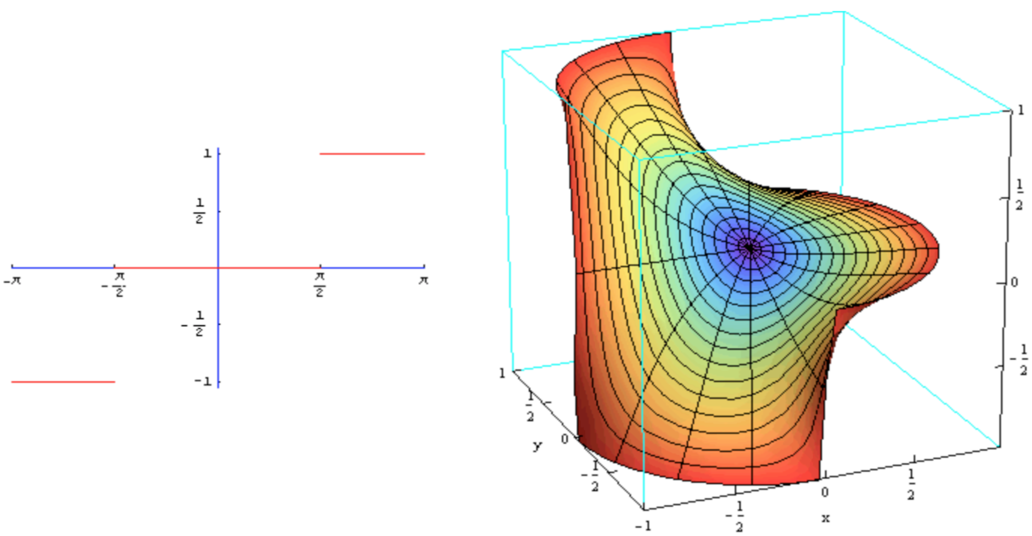


Figure 1. The graphs of $U(\theta) = u(\cos\theta, \sin\theta)$ and $u(x, y) = u(r\cos\theta, r\sin\theta)$.