Math 4200
Friday October 18
2.5 mean value property for analytic and harmonic functions, and maximum modulus principles. We'll begin on the page of Wednesday's notes we didn't get to. We'll use the maximum modulus principle in an interesting conformal mapping problem on Monday, which is related to the subject of the "hyperbolic disk" in hyperbolic geometry.

Announcements:

A cool way to justify the harmonic conjugate construction in simply connected domains....and also to deduce that harmonic functions are infinitely differentiable.....we appealed back to multivariable calculus earlier in the course for the conjugate construction. (Harmonic conjugates show up in our proof of the mean value property for harmonic functions, at the end of Wednesday's notes.)

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ be open and simply connected, and let $u : A \rightarrow \mathbb{R}$ be C^2 and harmonic. Then there exists a harmonic conjugate $v : A \rightarrow \mathbb{R}$, i.e. so that $f = u + i\,v$ is analytic. Furthermore, both u, v are actually C^∞, i.e. all partial derivatives exist and are continuous.

proof. If f existed, then f would be infinitely complex differentiable, and so in particular f' would be analytic...

$$\begin{aligned} f' &= f_x = u_x + i\,v_x \\ &= v_y - i\,u_y. \end{aligned}$$

In other words,

$$g(z) = u_x - i\,u_y$$

would be analytic. Actually, CR holds for $g(z)$ defined as above just because u is harmonic and C^2, and because g has continuous first partials, so g IS analytic: Check:

Since g is analytic on A and A is simply connected, g has an antiderivative $G = U + i\,V$. $G' = g$ so

$$U_x + i\,V_x = V_y - i\,U_y = u_x - i\,u_y$$

so $U_x = u_x$, $U_y = u_y$ so $U = u + C$ where C is a real constant because A is connected. Thus

$$f := G - C = u + i\,V$$

is analytic, i.e. V is a harmonic conjugate to u. Since G is infinitely complex differentiable, u, V are infinitely real differentiable.

QED.

Theorem (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f: A \to \mathbb{C}$ be analytic, $f: \overline{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \overline{A}}\{|f(z)|\} = \max_{z \in \delta A}\{|f(z)|\} := M,$$

i.e. the maximum modulus of $f(z)$ occurs on the boundary of A. (Recall that for an open set A, the boundary $\delta A = \overline{A} \setminus A$. For general sets the boundary is the collection of points in the closure of the set as well as in the closure of its complement.)

Furthermore if $\exists\, z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function on A.

Example: What is the maximum modulus of $f(z) = (z-2)^2$ on the closed disk $\overline{\mathbb{D}}(0;2)$ and where does it occur ?

proof of maximum modulus principle: Let

$$B = \{z \in A \mid |f(z)| = M\}$$

Our goal is to show that either:

(i) $B = \varnothing$, which implies that all points in $\overline{A}$ for which $|f(z)| = M$ are on the boundary of A, as the theorem claims. And in this case there is no $z_0 \in A$ with $|f(z_0)| = M$.

OR

(ii) $B = A$. In this case $|f(z)|$ is constant. And then we appeal to an old homework problem to conclude that actually $f(z)$ is constant: If $|f(z)|$ is constant in A, write $f = u + iv$ and so we have

$$u^2 + v^2 \equiv M^2$$

If $M = 0$ then $f = 0$ on A. If $M > 0$ then taking x and y partials we get the system for each $z \in A$:

$$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $M \neq 0$, $(u, v) \neq 0$. Thus the determinant of the matrix is zero. But the determinant of the matrix is

$$u_x v_y - u_y v_x = u_x^2 + u_y^2 = v_y^2 + v_x^2.$$

Thus the gradients of u, v are identically zero on the connected open set A, so u and v are each constants on A and f is as well. This must be the case that occurs if $\exists\, z_0 \in A$ with $|f(z_0)| = M$.

Following the outline on the previous page, we have

$$B = \{z \in A \mid \ |f(z)| = M\}.$$

Suppose we are not in case (i), i.e. $B \neq \varnothing$. We will show that B is open and closed in A which will imply that B must be all of A, since A is connected. Thus we are in case (ii).

Why is B closed in A?

To show B is open, let $z_0 \in B$, $D(z_0, \rho) \subseteq A$. We'll show $|f(z)| = M \ \forall\, z \in D(z_0, \rho)$. Each such z in the disk is of the form $z = z_0 + r\,e^{i\theta}$ with $r < \rho$. But for $0 < r < \rho$ we have the mean value property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + r\,e^{i\theta}\right)\,d\theta.$$

Use this and $|f(z_0)| = M$ to show each $\left|f\left(z_0 + r\,e^{i\theta}\right)\right| = M$ as well.

This generalizes an exercise you handed in on Wednesday, where you assumed a domain A was bounded by a simply connected, p.w. C^1 contour...

Theorem

Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f, g : A \rightarrow \mathbb{C}$ be analytic, $f, g : \overline{A} \rightarrow \mathbb{C}$ continuous. Then

$$\max_{z \in \overline{A}}\{|f(z) - g(z)|\} = \max_{z \in \delta A}\{|f(z) - g(z)|\}.$$

In particular, if $f = g$ on δA, then $f = g$ on all of A.

proof:

Theorem (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u : A \to \mathbb{R}$ be harmonic and C^2, $u : \overline{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x,y) \in \overline{A}} \{u(x,y)\} = \max_{(x,y) \in \delta A} \{u(x,y)\} := M,$$
$$\min_{(x,y) \in \overline{A}} \{u(x,y)\} = \min_{(x,y) \in \delta A} \{u(x,y)\} := m,$$

Furthermore if $\exists \; (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then u is a constant function on A.

Example: $u(x,y) = x^2 - y^2$ is harmonic. Where are the maximum and minimum values of u attained, on $\overline{D}(0; 2)$?

proof: The maximum principle implies the minimum principle, since the minimum principle for $u(x,y)$ is equivalent to the maximum principle for $v(x,y) = -u(x,y)$. In other words, minimum values for $u(x,y)$ correspond to maximum values for $-u(x,y)$, and u is harmonic if and only if $-u$ is. So we'll focus on the maximum principle. The key tool is the mean value principle for harmonic functions: For every closed disk in A, the average value of u on the bounding circle equals the value at the center. Can you see how the proof goes, if we follow the outline of the maximum modulus principle proof?

Use: if $D(z_0; \rho) \subseteq A$ then for each $0 < r < \rho$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) \, d\theta$$