Fundamental Theorem of Algebra Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

be a polynomial of degree n (scaled so that the coefficient of z^n is 1), with $a_j \in \mathbb{C}$. Then $p(z)$ factors into a product of linear factors,

$$p(z) = (z - z_1)(z - z_2) \ldots\ldots (z - z_n).$$

proof:

- It suffices to prove there exists a single linear factor when $n \geq 1$ since the general case then follows by induction:
  (i) The FTA is true when $n = 1$.
  (ii) If FTA is true for $n - 1$, and if
$$p_n(z) = (z - z_n)p_{n-1}(z)$$
then FTA is true for $p_n(z)$.

- To show that $p_n(z)$ has a linear factor, it suffices to show that $p_n(z)$ has a root, $p_n(z_n) = 0$. This follows from the division algorithm:

$$\cancel{\frac{p_n(z)}{z-a} = q_{n-1}(z) + \frac{R}{z-a}}$$

where R is the remainder. This can be rewritten as

$$p_n(z) = (z-a)q_{n-1}(z) + R.$$

So $p_n(a) = 0$ if and only if $(z - a)$ is a factor of $p_n(z)$.

use quotient rule!

Then the proof proceeds by contradiction: If $p_n(z)$ has no roots, then $\frac{1}{p_n(z)}$ is entire, and

$$\frac{1}{p_n(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} = \frac{1}{z^n} \underbrace{\frac{1}{\left(1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}}. \quad (z \neq 0)$$

Show that $\frac{1}{p_n(z)}$ must be bounded, so by Liouville's Theorem it must be constant. This is a contradiction!

- $\lim_{|z|\to\infty} \frac{1}{|p_n(z)|} = 0$ $\xrightarrow{\text{shortcut}}$ If believe standard limit theorems for sums & products of fcns, as $z \to \infty$

then $\lim_{|z|\to\infty}$ (1 above)

$= \left(\lim_{|z|\to\infty} \frac{1}{|z|^n}\right) \lim_{|z|\to\infty} \left|\frac{1}{1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}}\right|$

$= 0 \cdot 1 = 0$ ∎

(see text for ε, R proof)

So. Pick R s.t. $|z| \geq R \Rightarrow \frac{1}{|p_n(z)|} \leq 1$

But $\frac{1}{|p_n(z)|}$ is cont. on $\overline{D(0;R)}$ (since analytic)

↑ compact

so it attains its max "M" $\left(\frac{1}{|p_n(z)|} \leq M \;\; \forall z \in \overline{D(0;R)}\right)$.

$\Rightarrow \frac{1}{|p_n(z)|} \leq \max(M, 1)$ on $\mathbb{C}$ so $\frac{1}{p_n(z)}$ is bounded.

$\Rightarrow \frac{1}{p_n(z)} = C$ by Liouville! $(C \neq 0) \Rightarrow p_n(z) = \frac{1}{C}$ $\Rightarrow\!\Leftarrow$.

Math 4200
Wednesday October 16
2.4 Consequences of Cauchy's integral formula: The fundamental theorem of algebra; Morerra's theorem and uniform limits of analytic functions; mean value property for analytic and harmonic functions.

Announcements: We'll prove the fundamental theorem of algebra first (Monday's notes).

next HW assignment at end of notes

Estimates: We used a first derivative estimate via C.I.F. to prove Liouville's Theorem. Estimates for all derivatives are sometimes useful, and the most useful case is for the derivative estimate in the center of a disk.

Let $f: A \to \mathbb{C}$ analytic, (A open as always ... our running assumption on domains is that they are open connected sets, not necessarily simply connected though.) Let the closed disk $\overline{D}(z_0; R) \subseteq A$. Let γ be the circle of radius R, traversed once counterclocwise, so $I(\gamma; z_0) = 1$. Then we have the C.I.F and C.I.F. for derivatives,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} d\zeta$$

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$ exists

Let M be the maximum of $|f|$ on $\overline{D}(z_0; R)$, so also a bound for $|f|$ on γ. Then

$$\left| f'(z_0) \right| \le \frac{1}{2\pi} \int_\gamma \frac{|f(\zeta)|}{|\zeta - z_0|^2} |d\zeta| \le \frac{M}{2\pi} \frac{1}{R^2} 2\pi R = \frac{M}{R}.$$

$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{n!\, M}{R^n}.$$ $\leftarrow$ see w8.1

We used the first derivative estimate to prove Liouville's Theorem on Monday. You will have the opportunity to use the higher order derivative estimates in your homework this week.

we'll use the following result, and especially its corollary at key points of Chapter 3:

<u>Morera's Theorem</u> Let $f: A \to \mathbb{C}$ be continuous, and suppose the rectangle lemma holds, i.e.

$$\forall\ R = \{z = x + iy \mid a \le x \le b,\ c \le y \le d\} \subseteq A,$$
$$\int_{\delta R} f(z)\, dz = 0.$$

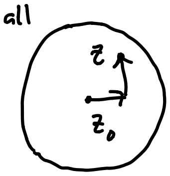
Then f is actually analytic on A.

proof: Let $z_0 \in A$. Let $D(z_0;2R) \subset A$ $\left(\Rightarrow \overline{D(z_0;R)} \subset A\right)$

rectangle lemma in $D(z_0;2R) \Rightarrow \exists F$ s.t. $F' = f$ in $D(z_0;2R)$

on $D(z_0;R)$ & $\gamma: |z - z_0| = R$.

(recall



$F(z) = \int_{\Gamma} f(\zeta)\, d\zeta$.)

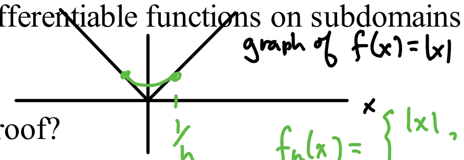
$$F^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

exists.

$\Rightarrow F''(z_0)$ exists. $(= f'(z_0))$.

Corollary Let $\{f_n\}: A \to \mathbb{C}$ analytic. Suppose $\{f_n\} \to f$ uniformly on A. Then $f: A \to \mathbb{C}$ is also analytic. (Contrast this with the analogous false theorem for differentiable functions on subdomains of $\mathbb{R}$).

This is so false in regular calculus:



$$f_n(x) = \begin{cases} |x|, & |x| \geq 1/n \\ \text{parabolic} & |z| \leq 1/n \end{cases}$$

proof: Can you check these peices, and combine them into a proof?

(i) f is continuous, because uniform limits of continuous functions are continuous. (3210-3220?)

(ii) If $\{f_n\} \to f$ uniformly on A and if the rectangle lemma holds for each f_n (which it does, because each f_n is analytic), then the rectangle lemma holds for f.

Let $\{f_n\} \to f$ uniformly. $\forall \varepsilon > 0 \ \exists N$ s.t. $|f_n(z) - f(z)| < \varepsilon/3 \ \forall n \geq N, z \in A$

(i) Let $z_0 \in A$, z near z_0

$$f(z) - f(z_0) = \overset{1}{\left(f(z) - f_N(z)\right)} + \overset{2}{\left(f_N(z) - f_N(z_0)\right)} + \overset{3}{\left(f_N(z_0) - f(z_0)\right)}$$

- f_N is cont. so $\exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f_N(z) - f_N(z_0)| < \varepsilon/3$

so $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| \leq |1| + |2| + |3| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

(ii) $\left\{ x + iy \text{ s.t. } \begin{matrix} a \leq x \leq b \\ c \leq y \leq d \end{matrix} \right\}$ = Rect. in A.

$\int_{\partial R} f_n(z)\,dz = 0$. prove $\int_{\partial R} f(z)\,dz = 0$. Then Morera $\Rightarrow f$ analytic.

Let $\varepsilon > 0$. use f_N.

$$\int_{\partial R} f(z)\,dz = \underbrace{\int_{\partial R} f_N(z)\,dz}_{0} + \int_{\partial R} f(z) - f_N(z)\,dz$$

$$\left| \int_{\partial R} f(z)\,dz \right| \leq \int_{\partial R} |f(z) - f_N(z)|\,|dz| \leq \frac{\varepsilon}{3}\left(2(b-a) + 2(d-c)\right)$$

holds $\forall \varepsilon > 0$ (use different N)

$$\Rightarrow \left| \int_{\partial R} f(z)\,dz \right| = 0.$$

One of the most-studied analytic functions is the *Riemann -Zeta function*. It is customary to write the complex variable as s in this case, rather than z. And for $\mathrm{Re}(s) > 1$, the Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where for $s = x + i\,y$, each term

$$n^{-s} = \mathrm{e}^{-s\log(n)} = \mathrm{e}^{-(x + i\,y)\ln(n)} = n^{-x}\mathrm{e}^{-i\,y\ln(n)}$$

is analytic in s. Note that for $x > 1$, the sum of moduli

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty}\frac{1}{n^x} < \infty$$

and for $x \geq 1 + \delta$ (with $\delta > 0$) the absolute convergence is uniform, so also the partial sums

$$\zeta_N(s) := \sum_{n=1}^{N}\frac{1}{n^s}$$

converge uniformly to $\zeta(s)$. Thus $\zeta(s)$ is analytic on the half plane $\mathrm{Re}(s) > 1$, by Morera's Theorem. Your favorite divergent series

$$\zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n} = +\infty$$

shows that $\zeta(s)$ is not analytic at $s = 1$. Somewhat surprisingly, $\zeta(s)$ can be extended to be analytic in all of $\mathbb{C} \setminus \{1\}$, however. (Such extensions must always be unique, it turns out.) The formulas for this extended function $\zeta(s)$ look different than the one that works on the half plane $\mathrm{Re}(s) > 1$.

The Riemann Zeta function has surprising connections to *number theory*, in particular to the *prime number theorem*, which is about how prime numbers are distributed in the natural numbers.

The *Riemann Hypothesis* is Riemann's conjecture from the 1800's, that all of the zeroes of the Riemann function lie on the line $\left\{\mathrm{Re}(s) = \frac{1}{2}\right\}$. It's considered one of the greatest unproven conjectures in mathematics, see for example the *Millenium prizes*. Of the billions of zeroes of the Riemann function which have been found, they're all on that line! Many results in number theory would follow if the Riemann hypothesis is true, so people are in the habit of proving theorems, where one of the assumptions is that the Riemann Hypothesis is true.

This is a great topic area for a research report in our course, if your interests go in this direction.

Mean value property (applications to follow on Friday): Let $f: A \rightarrow \mathbb{C}$ analytic, $\bar{D}(z_0; R) \subseteq A$. Then the value of f at z_0 is the average of the values of f on the concentric circle of radius R about z_0:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + R\,e^{i\theta}\right)\,d\theta$$

proof:

Remark harmonic functions also satsify a mean value property. How do you think you'd go about proving it?

Math 4200-001
Week 8 concepts and homework
2.4-2.5
Due Wednesday October 23 at start of class.

2.4 3, 5 (hint: identify the contour integrals as formulas for certain derivatives of analytic functions at certain points), 7, 8, 12, 16, 17, 18.
2.5 2, 5, 6, 7, 8, 15, 18.

w8.1 Let $f: \mathbb{C} \to \mathbb{C}$ be entire. Suppose that f grows at most like a power of z, as $z \to \infty$. In other words, suppose there exists $M, n \in \mathbb{N}$ such that for $|z| \geq 1$, $|f(z)| \leq M|z|^n$. Then prove that $f(z)$ must be a polynomial, and that its degree is at most n.