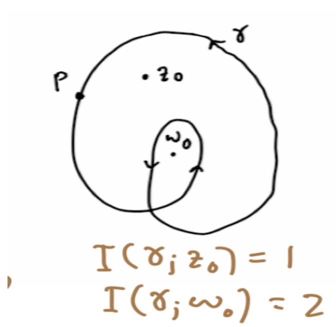
Math 4200
Monday October 14
2.4 Consequences of Cauchy's integral formula: infinite differentiability of analytic functions; Liouville's Theorem and the fundamental theorem of algebra.

Announcements:

2.4 Recall that the Friday before break we talked about winding number (index) of a closed curve about a point, and saw that it could be computed with a contour integral:

<u>Theorem</u> If γ is a piecewise C^1 closed contour and z_0 does not lie on (the image of) γ, then the index of γ with respect to z_0 can be computed with a contour integral:

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\, dz.$$



We used the contour integral formula for index to prove the <u>Cauchy Integral Formula</u> :

Let $A \subseteq \mathbb{C}$ be open
$f: A \to \mathbb{C}$ analytic
$\gamma: [a, b] \to \mathbb{C}$ a piecewise C^1 closed contour in A that is homotopic (as closed curves in A) to a point.
Let $z_0 \notin \gamma([a, b])$.
Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) I(\gamma; z_0).$$

(So, if z_0 is inside γ then $f(z_0)$ is determined and computable just from the values of f along γ !!!)

Note: Actually, as I stated the C.I.F. before break, I assumed that A was simply connected. This is not necessary as long as you assume that γ is homotopic to a point in A. Here's a review/sketch of the proof in this slightly more general case:

proof: Let

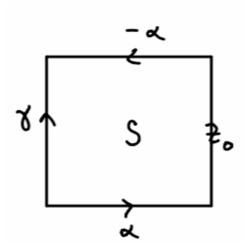
$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

1) g is analytic in $A \setminus \{z_0\}$ and continuous at z_0.

2) So the modified rectangle lemma holds (as we discussed before break).

3) So the local antiderivative theorem holds for $g(z)$.

4) Therefore the homotopy lemma holds for $g(z)$ and the homotopy $H(s, t)$ of γ to a point in A, through closed curves:



$$0 = \int_\alpha g(z)\, dz + \int_{z_0} g(z)\, dz - \int_\alpha g(z)\, dz - \int_\gamma g(z)\, dz$$

$$\Rightarrow \int_\gamma g(z)\, dz = 0\ .$$

5) Finally, since $z_0 \notin \gamma$,

$$0 = \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0)\int_\gamma \frac{1}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) 2\pi i\, I(\gamma; z_0)$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) I(\gamma; z_0)\ .$$

Q.E.D.

First application of C.I.F. : f analytic implies f is infinitely differentiable, with estimates for the moduli of the derivatives. Rewrite the CIF as

$$f(z)I(\gamma;z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

(we replaced z_0 by z, and the contour integral variable z by ζ.)

<u>Theorem 1</u> Let f be analytic in the open set $A \subseteq \mathbb{C}$, γ a p.w. C^1 contour homotopic to a point in A. Then for z inside γ, every derivative of f exists and may be computed by the contour integral formulas

$$f'(z)I(\gamma;z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f^{(n)}(z)I(\gamma;z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

notice, these are the formulas we get by induction and "differentiating thru the integral sign" :

$$\frac{d}{dz}\frac{f(\zeta)}{\zeta - z} = f(\zeta)(-1)(\zeta - z)^{-2}(-1) = \frac{f(\zeta)}{(\zeta - z)^2}$$

$$\frac{d}{dz}\frac{f(\zeta)}{(\zeta - z)^n} = f(\zeta)(-n)(\zeta - z)^{-n-1}(-1) = n\frac{f(\zeta)}{(\zeta - z)^{n+1}}.$$

So, when can you justify this operation of differentiating thru the integral sign? That's an analysis question!

Analysis answer! General setup: Let γ as usual and

$$G(z) := \int_\gamma g(z,\zeta)\, d\zeta.$$

(For our current needs we will be using the special cases

$$g(z,\zeta) = \frac{f(\zeta)}{(\zeta - z)^n})$$

By linearity of integration,

$$\frac{G(z+h) - G(z)}{h} = \int_\gamma \frac{g(z+h,\zeta) - g(z,\zeta)}{h}\, d\zeta.$$

We wish to know general conditions under which these contour integrals of difference quotients converge to

$$\int_\gamma \frac{\partial}{\partial z} g(z,\zeta)\, d\zeta$$

as $h \to 0$. We certainly need that $g(z,\zeta)$ be complex differentiable in the z variable. Then the following suffices: Suppose the difference quotients converge uniformly (with respect to $\zeta \in \gamma[a,b]$) to $\frac{\partial}{\partial z} g(z,\zeta)$. In other words,

$$\forall\ \varepsilon > 0\ \exists\ \delta > 0 \text{ such that } \forall\ \zeta \in \gamma[a,b]$$

$$|h| < \delta \Rightarrow \left| \frac{g(z+h,\zeta) - g(z,\zeta)}{h} - \frac{\partial}{\partial z} g(z,\zeta) \right| < \varepsilon.$$

If this uniformity condition holds, then

$$|h| < \delta \Rightarrow \left| \frac{G(z+h) - G(z)}{h} - \int_\gamma \frac{\partial}{\partial z} g(z,\zeta)\, d\zeta \right|$$

$$\leq \int_\gamma \left| \frac{g(z+h,\zeta) - g(z,\zeta)}{h} - \frac{\partial}{\partial z} g(z,\zeta) \right| |d\zeta| < \varepsilon \cdot \text{length}(\gamma),$$

which implies

$$G'(z) = \int_\gamma \frac{\partial}{\partial z} g(z,\zeta)\, d\zeta .$$

So, when can we verify the uniformity condition from the previous page?

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \forall\, \zeta \in \gamma[a,b]$$

$$|h| < \delta \Rightarrow \left| \frac{g(z+h,\zeta) - g(z,\zeta)}{h} - \frac{\partial}{\partial z} g(z,\zeta) \right| < \varepsilon.$$

estimate, assuming $g(z,\zeta)$ is analytic in the z- variable and using e.g. line segment contours from z to $z+h$:

$$\frac{g(z+h,\zeta) - g(z,\zeta)}{h} = \frac{1}{h} \int_{z \to z+h} \frac{\partial}{\partial w} g(w,\zeta)\, dw$$

$$= \frac{1}{h} \int_{z \to z+h} \frac{\partial}{\partial z} g(z,\zeta) + \left(\frac{\partial}{\partial w} g(w,\zeta) - \frac{\partial}{\partial z} g(z,\zeta) \right) dw$$

$$= \frac{h}{h} \frac{\partial}{\partial z} g(z,\zeta) + \frac{1}{h} \int_{z \to z+h} \left(\frac{\partial}{\partial w} g(w,\zeta) - \frac{\partial}{\partial z} g(z,\zeta) \right) dw.$$

Regarding the second term as the error term: If for sufficiently small $\rho > 0$, $\frac{\partial}{\partial w} g(w,\zeta)$ is continuous for $(w,\zeta) \in \bar{D}(z;\rho) \times \gamma([a,b])$, then it is uniformly continuous, so

$$\forall\, \varepsilon > 0\ \exists\, 0 < \delta < \rho \text{ such that } \forall\, \zeta \in \gamma[a,b], |w - z| < \delta,$$

$$\left| \frac{\partial}{\partial w} g(w,\zeta) - \frac{\partial}{\partial z} g(z,\zeta) \right| < \varepsilon.$$

And in this case, for $|h| < \delta$, the error term is bounded uniformly for $\zeta \in \gamma[a,b]$, by

$$\left| \frac{1}{h} \int_{z \to z+h} \left(\frac{\partial}{\partial w} g(w,\zeta) - \frac{\partial}{\partial z} g(z,\zeta) \right) dw \right| \le \left| \frac{h}{h} \right| \varepsilon = \varepsilon .$$

In our applications for the Cauchy integral formulas for derivatives,

$$g(z,\zeta) = \frac{f(\zeta)}{(\zeta - z)^n}$$

$$\frac{\partial}{\partial w} g(w,\zeta) = \frac{n f(\zeta)}{(\zeta - w)^{n+1}}$$

is continuous for $(w,\zeta) \in \bar{D}(z;\rho) \times \gamma([a,b])$ as soon as ρ is small enough so that $\bar{D}(z;\rho) \times \gamma([a,b]) = \varnothing$. (Positive distance lemma).

This finishes the analysis explanation for why the Cauchy integral formulas for derivatives hold.

Liouville's Theorem Let $f : \mathbb{C} \to \mathbb{C}$ be entire and bounded. Then f is constant!
proof: Use the Cauchy integral formula for $f'(z)$ on disks or radius R centered at z, as $R \to \infty$, to show $f'(z) = 0 \;\; \forall\, z \in \mathbb{C}$.

Fundamental Theorem of Algebra Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

be a polynomial of degree n (scaled so that the coefficient of z^n is 1), with $a_j \in \mathbb{C}$. Then $p(z)$ factors into a product of linear factors,

$$p(z) = (z - z_1)(z - z_2) \ldots\ldots (z - z_n).$$

proof:

- It suffices to prove there exists a single linear factor when $n \geq 1$ since the general case then follows by induction:
  (i) The FTA is true when $n = 1$.
  (ii) If FTA is true for $n - 1$, and if

$$p_n(z) = (z - z_n)p_{n-1}(z)$$

then FTA is true for $p_n(z)$.

- To show that $p_n(z)$ has a linear factor, it suffices to show that $p_n(z)$ has a root, $p_n(z_n) = 0$. This follows from the division algorithm:

$$\frac{p_n(z)}{z - a} = q_{n-1}(z) + \frac{R}{z - a}$$

where R is the remainder. This can be rewritten as

$$p_n(z) = (z - a)q_{n-1}(z) + R.$$

So $p_n(a) = 0$ if and only if $(z - a)$ is a factor of $p_n(z)$.

Then the proof proceeds by contradiction: If $p_n(z)$ has no roots, then $\dfrac{1}{p_n(z)}$ is entire, and

$$\frac{1}{p_n(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} = \frac{1}{z^n} \frac{1}{\left(1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}.$$

Show that $\dfrac{1}{p_n(z)}$ must be bounded, so by Liouville's Theorem it must be constant. This is a contradiction!