We have space to work the table entries (4) and (6) on Monday's notes. There is also the sledgehammer Proposition 4.1.7 which might be a fun extra credit homework problem, but which I've almost never found an occassion to actually use beyond the easier table entries beneath it. It follows by systematically extending the method of (6).

If you have occassion to use one of the table entries and you haven't checked it before, you might want to verify it first. Some of these are easy pretty easy:

7) If $f(z)$ has a pole of order k at z_0, and has the form

$$f(z) = \frac{g(z)}{(z-z_0)^k} = \frac{\sum_{n=0}^{\infty} a_n (z-z_0)^n}{(z-z_0)^k} = \sum_{n=0}^{\infty} a_n \frac{(z-z_0)^n}{(z-z_0)^k}$$

$\frac{1}{z-z_0}$ when $n = k-1$

then

$$Res(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

$a_{k-1} = \text{res} = \frac{f^{(k-1)}(z_0)}{(k-1)!}$

ex $Res\left(\frac{e^z}{z^3}; 0\right)$

$$= \frac{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z} + \cdots$$

10) If $f(z)$ has a pole of order k at z_0, with

$$f(z) = \frac{g(z)}{h(z)}$$

where g has a zero of order l and $h(z)$ has a zero of order $k+l$, then for the function

$$\phi(z) = (z-z_0)^k f(z),$$

which has a removable singularity at z_0,

$$Res(f, z_0) = \lim_{z \to z_0} \frac{\phi^{(k-1)}(z)}{(k-1)!}$$

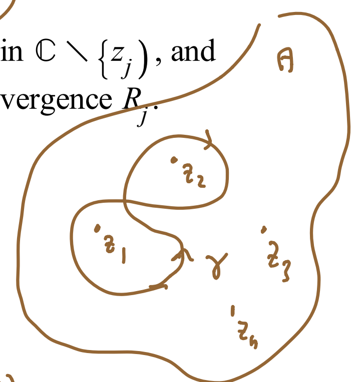
Residue Theorem (Deformation Theorem version, more general than the Green's Theorem version.). Let Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \ldots z_k\} \subseteq A$. Let γ be a closed curve which is homotopic to a point in A. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{k} Res(f, z_j)\, I(\gamma; z_j)$$

Being more general than the Green's Theorem version, this proof is also a bit more ~~complicated~~ slick. For each isolated singularity z_j we have the Laurent series

$$S_j(z) = S_{j1}(z) + S_{j2}(z) = \sum_{n=0}^{\infty} a_{jn}(z-z_j)^n + \sum_{m=1}^{\infty} \frac{b_{jm}}{(z-z_j)^m}.$$

Because the z_j are point singularities, the singular part of the series, $S_{j2}(z)$ converges in $\mathbb{C} \setminus \{z_j\}$, and the non-singular part converges for $0 \le |z - z_j| < R_j$ for some positive radius of convergence R_j.



Now consider

$$g(z) := f(z) - \sum_{j=1}^{k} S_{j2}(z).$$

Explain why $g(z)$ has removable singularities at each z_l:

near z_ℓ, $g(z) = \underbrace{f(z) - S_{\ell 2}(z)}\; - \underbrace{\sum_{j \neq \ell} S_{j2}(z)}_{\text{analytic near } z_\ell}$

$- S_{\ell 2}(z)$

$= \sum_{n=0}^{\infty} a_{\ell n}(z - z_\ell)^n$ also analytic

Thus we may consider g to be analytic in A, so since γ is homotopic as closed curves to a point in A,

$$\int_\gamma g(z)\, dz = 0.$$

Expand this to get the result!

$$\int_\gamma f(z) - \sum_{j=1}^{k} S_{j2}(z)\; dz = 0$$

$$\int_\gamma f(z)\,dz = \int_\gamma \sum_{j=1}^{k} \sum_{m=1}^{\infty} b_{jm}/(z-z_j)^n \; dz$$

interchange $\int$ with $\sum\sum$

as always only $\frac{1}{z-z_j}$ terms have non-zero int

$$= \sum_{j=1}^{n} \underbrace{b_{j1}}_{Res(f, z_j)} \underbrace{\int_\gamma \frac{1}{z-z_j}\, dz}_{2\pi i\, I(\gamma; z_j)}$$

Math 4200
Friday November 8

4.1-4.2 The Residue Theorems and residue table entries.

Announcements: We'll start by finishing Wednesday's notes

There was a table entry we didn't get to in Monday's notes, #6, that's kind of fun and hints at the general formula Proposition 4.1.7, which is an extra credit problem in this week's homework.

6) Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$. Then f has a pole of order 2 and

$$Res(f, z_0) = \frac{2\,g'(z_0)}{h''(z_0)} - \frac{2}{3}\,\frac{g(z_0)h'''(z_0)}{h''(z_0)^2} \qquad !!!$$

$$f(z) = \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}{b_2(z-z_0)^2 + b_3(z-z_0)^3 + \dots} = \frac{a_0 + a_1(z-z_0) + \dots}{(z-z_0)^2\left[b_2 + b_3(z-z_0) + \dots\right]}$$

pole of order 2.

$$\left(\frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + \dots\right) = \frac{1}{(z-z_0)^2}\left[\text{———}\right]$$

mult by denoms on right

$$\left(c_{-2} + c_{-1}(z-z_0) + c_0(z-z_0)^2 + \dots\right)\left(b_2 + b_3(z-z_0) + \dots\right) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

1: $\quad c_{-2}b_2 = a_0$

$(z-z_0)$: $\quad c_{-1}b_2 + c_{-2}b_3 = a_1$

2270 more syst.

$$\begin{bmatrix} b_2 & 0 \\ b_3 & b_2 \end{bmatrix}\begin{bmatrix} c_{-2} \\ c_{-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Use Cramer's rule to get same ans.

$$c_{-2} = \frac{a_0}{b_2}$$

$$c_{-1}b_2 + \frac{a_0}{b_2}b_3 = a_1 \Rightarrow c_{-1} = \frac{a_1}{b_2} - \frac{a_0}{b_2^2}b_3 = \frac{g'(z_0)}{\frac{1}{2}h''(z_0)} - \frac{g(z_0)\frac{1}{6}h'''(z_0)}{\left(\frac{1}{2}h''(z_0)\right)^2} \qquad \text{ans!!}$$

Example: Find

$$Res\left(\frac{e^z}{\cos(z) - 1}; 0\right)$$

$g(z) = e^z$

$h(z) = \cos z - 1.$

$z_0 = 0$

$$\| $$

$$\frac{2e^0}{-\cos(0)} - \frac{e^0 \cdot 0}{\sim} = -2$$

<u>Residue Theorem for exterior domains</u> (This is our 3rd residue theorem). Let γ be a simple closed contour enclosing a region A, oriented counterclockwise as usual. Let K be a compact subset in A (possibly empty), and let $\{z_1, z_2, \ldots z_n\}$ be points exterior to γ. Let

$$f : \mathbb{C} \setminus \{K \cup \{z_1, z_2, \ldots z_n\}\} \to \mathbb{C}$$

be analytic. Then

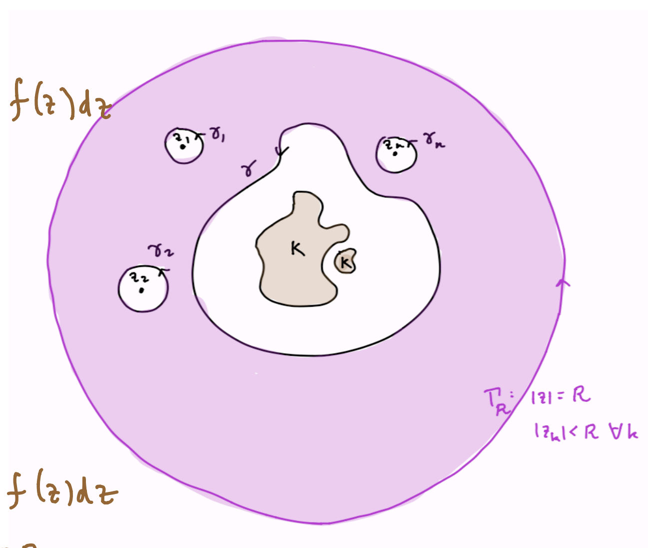
$$\int_\gamma f(z)\,dz = -2\pi i \left(Res(f; \infty) + \sum_{k=1}^{n} Res(f; z_k) \right)$$

where

$$Res(f; \infty) := Res\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right); 0 \right).$$

proof: Enclose γ and all of the singularities $\{z_1, z_2, \ldots z_n\}$ in a large disk of radius R centered at the origin as pictured, and let γ_j be a circle concentric with the singularity z_j and of sufficiently small radius r_j so that the closed disks they enclose are disjoint, and don't intersect γ or Γ_R, the circle of radius R centered at the origin. Orient all curves counterclockwise. Then apply Cauchy's Theorem for domains with holes.

$$\oint_{|z|=R} f(z)\,dz = \sum_{j=1}^{n} \underbrace{\oint_{\gamma_j} f(z)\,dz}_{\substack{\text{old res. thm}\\ 2\pi i\, Res(f;\, z_j)}} + \oint_{\gamma} f(z)\,dz$$



$$\oint_{\gamma} f(z)\,dz = -2\pi i \sum_{j=1}^{n} Res(f; z_j) + \oint_{|z|=R} f(z)\,dz$$

You will arrive at an equation which is equivalent to

$$\int_\gamma f(z)\,dz = -2\pi i \sum_{k=1}^{n} Res(f; z_k) + \oint_{\Gamma_R} f(z)\,dz. \quad = -2\pi i\, Res(f(z); \infty)$$

To evaluate the contour integral over Γ_R do an analytic change of variables, $\zeta = \frac{1}{z}$, $z = \frac{1}{\zeta}$ which will give you a contour integral over a circle of radius $\frac{1}{R}$, traversed clockwise. Evaluate this integral with version 2 of the Residue Theorem and the result will follow. Analytic change of variables for contour integrals is justified on the next page.

$z = Re^{it}$, $0 \le t \le 2\pi$

$\zeta = \frac{1}{z} = \frac{1}{R} e^{-it}$

clockwise!

$$\oint_{|z|=R} f(z)\,dz = \oint_{|\zeta| = \frac{1}{R}} f\left(\frac{1}{\zeta}\right)\left(-\frac{1}{\zeta^2}\right) d\zeta$$

only sing here @ $\zeta = 0$

f no sing for $|z| \ge R$ $\left(\left|\frac{1}{z}\right| \le \frac{1}{R}\right)$

$$= 2\pi i\, Res\left(-\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right); 0\right)(-1)$$

index!

<u>Theorem</u> Analytic change of variables in contour integrals: Consider the contour integral

$$\int_\gamma f(z)\,dz.$$

Suppose there is an invertible analytic function g with range that includes γ, $z = g(\zeta)$, $\zeta = g^{-1}(z)$. Then the formal substitution $z = g(\zeta)$, $dz = g'(\zeta)\,d\zeta$ yields an equality of integrals

$$\int_\gamma f(z)\,dz = \int_{g^{-1}(\gamma)} f(g(\zeta))\; g'(\zeta)\; d\zeta.$$

proof: Let $\gamma : [a, b] \to \mathbb{C}$ be a parameterization of the contour on the left. Write $\varphi(t) = g^{-1}(\gamma(t))$ to parameterize the contour on the right. (Assume γ is C^1 rather than piecewise C^1 for simplicity). Compute both contour integrals and use the chain rule for curves to verify that the integrals agree.

⊙ this is checkable.

Example of Residue Theorem for exterior domains: Compute

$$\oint_\gamma \frac{3\,z^2+7}{z^3+2\,z-3}\,dz \quad = -2\pi i\, Res(f;\infty) = 6\pi i$$

where γ is the circle $|z| = 2$, oriented counter-clockwise as usual. First verify that all of roots of the cubic denominator lie inside the circle, so we'll only need the residue at ∞,

$$Res(f;\infty) := Res\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right);0\right).$$

$$(z-1)(z^2+z+3)$$

$$z = \frac{-1 \pm \sqrt{1-12}}{2}$$

$$|z|<2.$$

If $|z| \geq 2 \Rightarrow |z^3+2z-3| \geq |z^3| - |2z| - 3 \geq 8 - *$

OOPs!

but it's true.

correct reason: $* = |z|^3 - 2|z| - 3$

$p(t) = t^3 - 2t - 3$

$p(2) = 8-4-3 = 1 > 0$

$p'(t) = 3t^2 - 2 \geq 10$ for $t \geq 2$

$\Rightarrow p(t) > 0$ for $t \geq 2$.

$$-\frac{1}{z^2} f\left(\frac{1}{z}\right) =$$

$$-\frac{1}{z^2}\left[\frac{\frac{3}{z^2}+7}{\frac{1}{z^3}+\frac{2}{z}-3}\right]\frac{z^3}{z^3}$$

$$= -\frac{1}{z^2}\left[\frac{3z+7z^3}{1+2z^2-3z^3}\right]$$

$$= -\frac{1}{z}\left[\frac{3+7z^2}{1+2z^2-3z^3}\right]$$

$$Res(\%;0) = -3.$$