Math 4200
Friday November 8

4.1-4.2 The Residue Theorems and residue table entries.

Announcements: We'll start by finishing Wednesday's notes

There was a table entry we didn't get to in Monday's notes, #6, that's kind of fun and hints at the general formula Proposition 4.1.7, which is an extra credit problem in this week's homework.

6) Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$. Then f has a pole of order 2 and

$$Res(f, z_0) = \frac{2\, g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0) h'''(z_0)}{h''(z_0)^2} \qquad !!!$$

Example: Find

$$Res\left(\frac{e^z}{\cos(z) - 1}; 0\right)$$

Residue Theorem for exterior domains (This is our 3rd residue theorem). Let γ be a simple closed contour enclosing a region A, oriented counterclockwise as usual. Let K be a compact subset in A (possibly empty), and let $\{z_1, z_2, \dots z_n\}$ be points exterior to γ. Let

$$f: \mathbb{C} \setminus \{K \cup \{z_1, z_2, \dots z_n\}\} \to \mathbb{C}$$

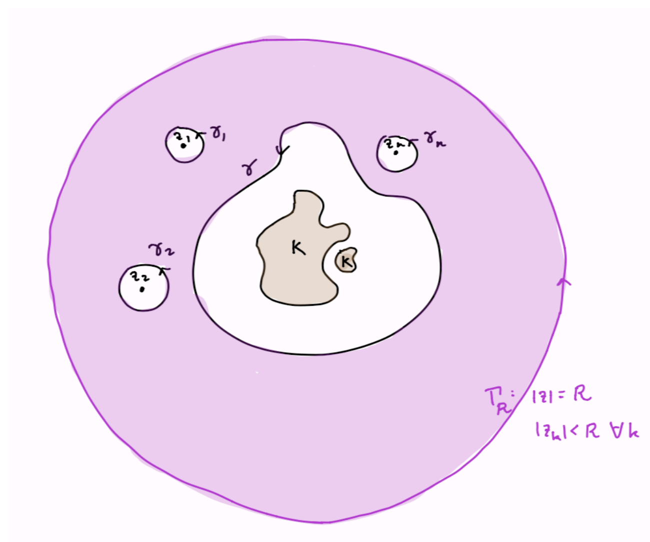
be analytic. Then

$$\int_{\gamma} f(z)\,dz = -2\pi i\left(Res(f; \infty) + \sum_{k=1}^{n} Res\left(f; z_k\right)\right)$$

where

$$Res(f; \infty) := Res\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right).$$

proof: Enclose γ and all of the singularities $\{z_1, z_2, \dots z_n\}$ in a large disk of radius R centered at the origin as pictured, and let γ_j be a circle concentric with the singularity z_j and of sufficiently small radius r_j so that the closed disks they enclose are disjoint, and don't intersect γ or Γ_R, the circle of radius R centered at the origin. Orient all curves counterclockwise. Then apply Cauchy's Theorem for domains with holes.



You will arrive at an equation which is equivalent to

$$\int_{\gamma} f(z)\,dz = -2\pi i \sum_{k=1}^{n} Res\left(f; z_k\right) + \int_{\Gamma_R} f(z)\,dz.$$

To evaluate the contour integral over Γ_R do an analytic change of variables, $\zeta = \frac{1}{z}$, $z = \frac{1}{\zeta}$ which will give you a contour integral over a circle of radius $\frac{1}{R}$, traversed clockwise. Evaluate this integral with version 2 of the Residue Theorem and the result will follow. Analytic change of variables for contour integrals is justified on the next page.

<u>Theorem</u> Analytic change of variables in contour integrals: Consider the contour integral

$$\int_\gamma f(z)dz.$$

Suppose there is an invertible analytic function g with range that includes γ, $z = g(\zeta)$, $\zeta = g^{-1}(z)$. Then the formal substitution $z = g(\zeta)$, $dz = g'(\zeta)d\zeta$ yields an equality of integrals

$$\int_\gamma f(z)dz = \int_{g^{-1}(\gamma)} f(g(\zeta))\, g'(\zeta)\, d\zeta.$$

proof: Let $\gamma : [a, b] \to \mathbb{C}$ be a parameterization of the contour on the left. Write $\varphi(t) = g^{-1}(\gamma(t))$ to parameterize the contour on the right. (Assume γ is C^1 rather than piecewise C^1 for simplicity). Compute both contour integrals and use the chain rule for curves to verify that the integrals agree.

Example of Residue Theorem for exterior domains: Compute

$$\int_{\gamma} \frac{3\,z^2 + 7}{z^3 + 2\,z - 3}\, dz$$

where γ is the circle $|z| = 2$, oriented counter-clockwise as usual. First verify that all of roots of the cubic denominator lie inside the circle, so we'll only need the residue at ∞,

$$Res\,(f;\,\infty\,) := Res\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right);\,0\right).$$