Computing residues In Chapter 4 we'll see lots of examples where we need to compute residues at isolated singularities z_0, because they are related to interesting contour integrals or to improper integrals on the real line. In fact section 4.1 is all about residue computation shortcuts in case the situation is complicated. (The residue computations in the homework for this Wednesday are mostly somewhat straightforward.)

In most cases the function of concern is a quotient and the reason for the singularity is that the denominator function has a zero at z_0. There's a scary-looking table on page 250 of our text - that will be provided on the next midterm - although I could ask you to verify certain entries in addition to using them. Here are two table entries, one easy, one a bit more complicated. We have everything we need to check these table entries, because we know how to multiply Taylor series and Laurent series:

1) Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$. Prove that f has a pole of order 1, and

#4

$$Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

$g(z) = g(z_0) + g'(z_0)(z-z_0) + \cdots$

$h(z) = \qquad h'(z_0)(z-z_0) + \cdots$

$$f(z) = \frac{g(z_0) + g'(z_0)(z-z_0) + \cdots}{(z-z_0)\left[h'(z_0) + \frac{1}{2!}h''(z_0)(z-z_0) + \cdots\right]}$$

$$f(z) = \frac{1}{z-z_0}\Bigg[\underbrace{\frac{g(z_0) + g'(z_0)(z-z_0) + \cdots}{h'(z_0) + \frac{1}{2!}h''(z_0)(z-z_0) + \cdots}}_{\text{analytic near } z_0}\Bigg]$$

$$= \frac{1}{z-z_0}\left[d_1 + d_2(z-z_0) + \cdots\right]$$

$$Res(f; z_0) = d_1 = \frac{g(z_0)}{h'(z_0)} \checkmark$$

2) Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0, h(z_0) = h'(z_0) = 0, h''(z_0) \neq 0$. Then f has a pole of order 2 and

#6

$$Res(f, z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{h''(z_0)^2} \qquad !!!$$

Math 4200
Wednesday November 6

4.1 Calculating residues at isolated singularities
4.2 The Residue Theorem(s)

Announcements:

- I'll return HW 9 by Fri. & HW10 by Monday
  HW 11 is at the end of today's notes
- exam will cover thru 4.2 (HW 11).
- review session Monday 4-5:30
  (like last time; room TBA)

Prop. 4.1.7: Let g, h analytic at z_0, $g(z_0) \neq 0$ and assume $h(z_0) = h'(z_0) = \ldots = h^{(k-1)}(z_0)$ and $h^{(k)}(z_0) \neq 0$.

Then g/h has a pole of order k and *the residue at z_0, $\mathrm{Res}(g/h; z_0)$ is given by*

$$\mathrm{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times \begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ matrix.

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

	Function	Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)\neq 0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=0$, $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$, $h(z_0)=0=h'(z_0)$, $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0)=0, g'(z_0)\neq 0$, $h(z_0)=0=h'(z_0)$ $=h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)}-\frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z)=(z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z)=(z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=$ $\ldots=h^{k-1}(z_0)$ $=0, h^k(z_0)\neq 0$	pole of order k	see Proposition 4.1.7.

We have space to work the table entries (4) and (6) on Monday's notes. There is also the sledgehammer Proposition 4.1.7 which might be a fun extra credit homework problem, but which I've almost never found an occassion to actually use beyond the easier table entries beneath it. It follows by systematically extending the method of (6).

If you have occassion to use one of the table entries and you haven't checked it before, you might want to verify it first. Some of these are easy pretty easy:

7) If $f(z)$ has a pole of order k at z_0, and has the form

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

then

$$Res(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

10) If $f(z)$ has a pole of order k at z_0, with

$$f(z) = \frac{g(z)}{h(z)}$$

where g has a zero of order l and $h(z)$ has a zero of order $k + l$, then for the function

$$\phi(z) = (z - z_0)^k f(z),$$

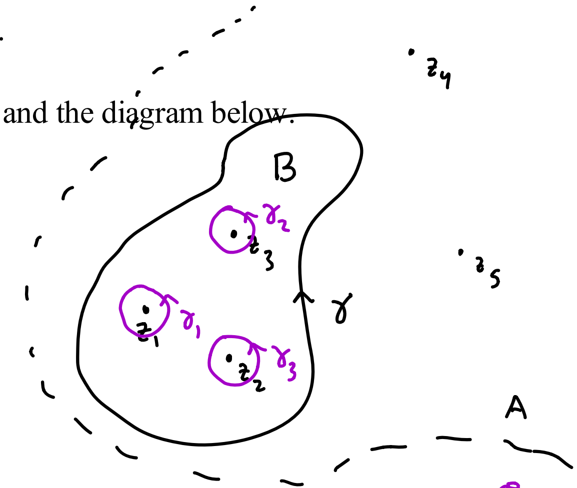
which has a removable singularity at z_0,

$$Res(f, z_0) = \lim_{z \to z_0} \frac{\phi^{(k-1)}(z)}{(k-1)!}$$

Residue Theorem (Green's Theorem version): Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \dots z_k\} \subseteq A$ Let γ be a simple closed contour in A which contains none of the singularities, and which bounds a a subregion B containing some of the singularities. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \in B} Res(f, z_j).$$

proof: Use Cauchy's Theorem for domains with holes, Laurent series, and the diagram below.



pick ε_j so that $\overline{D(z_j, \varepsilon_j)} \subset B$ for each $z_j \in B$

Let γ_j be $|z - z_j| = \varepsilon_j$, & so disks are disjoint

$$\oint_\gamma f(z)\,dz = \sum_{z_j \in B} \oint_{\gamma_j} f(z)\,dz \qquad (1)$$

Let $f(z) = S_{j1}(z) + S_{j2}(z)$ be L.S. for f near $z_j \in B$

$$= \underbrace{\sum_{n=0}^{\infty} a_{jn}(z-z_j)^n}_{\text{conv. on some disk centered @ } z_j} + \underbrace{\sum_{m=1}^{\infty} b_{jm}\frac{1}{(z-z_j)^m}}_{\text{conv on } \mathbb{C}\setminus\{z_j\}}$$

(1): $\int_{\gamma_j} f(z)\,dz = \int_{\gamma_j} \sum_{n=0}^{\infty} + \sum_{m=1}^{\infty} dz$

$$= \sum_{n=0}^{\infty} \underbrace{\int_{\gamma_j} - \,dz}_{=0 \text{ FTC}} + \sum_{m=1}^{\infty} \underbrace{\int_{\gamma_j} - \,dz}_{=0 \text{ FTC}}$$

except when m = 1

$$b_{j1} \underbrace{\int_{\gamma_j} \frac{1}{z - z_j}\,dz}_{2\pi i}$$

$2\pi i \, Res(f; z_j)$

$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{z_j \in B} 2\pi i \,(Res(f; z_j))$$

∎

Exercise: Show - just for the practice - that this theorem includes as special cases

(a) Green's Theorem version of Cauchy's theorem ($\int_\gamma f(z)\,dz = 0$ if f is analytic in A.)

(RHS = 0).

(b) CIF $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz$ if z_0 is inside γ and f is analytic in A.

use residue thm for $g(z) = \frac{f(z)}{(z - z_0)}$

single singular pt @ z_0

$Res(g; z_0)$

$$g(z) = \frac{f(z)}{z - z_0} = \frac{f(z_0) + f'(z_0)(z - z_0) + \cdots}{z - z_0}$$

$$= \frac{f(z_0)}{z - z_0} + f'(z_0) + \cdots$$

$Res(g; z_0) = f(z_0)$.

Res thm $\Rightarrow \int_\gamma g(z)\,dz = \int_\gamma \frac{f(z)}{z - z_0}\,dz = 2\pi i \, Res(g; z_0) = 2\pi i \, f(z_0)$

Table entry simple pole $f(z) = \frac{g(z)}{h(z)} = \frac{1}{z^2-1} = \frac{g(z)}{h(z)}$

$Res(f;z_0) = \frac{g(z_0)}{h'(z_0)}$ $\qquad Res(f;1) = \frac{1}{2\cdot 1} = \frac{1}{2}$

$Res(f;-1) = -\frac{1}{2}$

Pretty much all of our previous examples of contour integration can be computed via the Residue Theorem, and it's usually the quickest way. New examples become more straightforward.

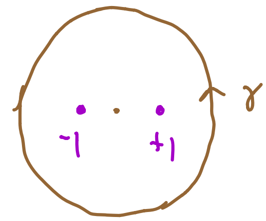
1) Compute

$$2\pi i\left(\frac{1}{2}-\frac{1}{2}\right) = 2\pi i\left(Res(f;1) + Res(f;-1)\right) = \int_{|\zeta|=2} \frac{1}{\zeta^2-1}\,d\zeta$$

$= 0$.

$f(z) = \frac{1}{z^2-1}$

singularities $z = \pm 1$



OR

$= \int_{|\zeta|=2} \frac{1}{2}\left(\frac{1}{\zeta-1} - \frac{1}{\zeta+1}\right) d\zeta$

$= \frac{1}{2}(2\pi i - 2\pi i)$

Can also read of the 2 residues from partial fracs.

$Res(f;1)$: $\frac{1}{z^2-1} = \frac{1}{z-1}\cdot\underbrace{\frac{1}{z+1}}_{g(z) \text{ analytic near } 1}$

$= \frac{1}{z-1}\left(g(1) + g'(1)(z-1) + \dots\right)$

$Res(f;1) = g(1) = \frac{1}{2}$

$Res(f;-1)$: $\frac{1}{z^2-1} = \frac{1}{z+1}\left(\frac{1}{z-1}\right)$ $\quad g(z)$ analytic near -1

$= \frac{1}{z+1}\left(g(-1) + g'(-1)(z+1) + \dots\right)$

$Res(f;-1) = g(-1) = -\frac{1}{2}$

2) Compute

$$2\pi i\left(\frac{e}{2} - \frac{1}{2e}\right) = 2\pi i\left(Res(f;1) + Res(f;-1)\right) = \int_{|\zeta|=2} \underbrace{\frac{e^\zeta}{\zeta^2-1}}_{f(\zeta)}\,d\zeta$$

$Res(f;1)$: $\frac{e^z}{z^2-1} = \frac{1}{z-1}\left[\frac{e^z}{z+1}\right]$

$Res(f;1) = \frac{e^z}{z+1}\Big|_{z=1}$ const term in taylor series for $\overbrace{\frac{e^z}{z+1}}^{g(z)} = \frac{e^1}{1+1} + g'(1)(z-1) + \dots$

$= \frac{e}{2}$

$Res(f;-1)$: $\frac{e^z}{z^2-1} = \frac{1}{z+1}\left[\frac{e^z}{z-1}\right]$

$= \frac{e^z}{z-1}\Big|_{z=-1} = \frac{e^{-1}}{-2}$

Math 4200-001
Week 11 concepts and homework
4.1-4.2
Due Wednesday November 13 at start of class.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 8, 9, 13, 15.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)