Math 4200
Wednesday November 6

4.1 Calculating residues at isolated singularities
4.2 The Residue Theorem(s)

Announcements:

Prop. 4.1.7: Let g, h analytic at z_0, $g(z_0) \neq 0$ and assume $h(z_0) = h'(z_0) = \ldots = h^{(k-1)}(z_0)$ and $h^{(k)}(z_0) \neq 0$.

Then g/h has a pole of order k and

the residue at z_0, $\mathrm{Res}(g/h; z_0)$ *is given by*

$$\mathrm{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times$$
$$\begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ *matrix.*

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

	Function	Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=0,$ $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0)=0=h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0)=0, g'(z_0)\neq 0,$ $h(z_0)=0=h'(z_0)$ $=h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)}-\frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z)=(z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z)=(z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=$ $\ldots=h^{k-1}(z_0)$ $=0, h^k(z_0)\neq 0$	pole of order k	see Proposition 4.1.7.

We have space to work the table entries (4) and (6) on Monday's notes. There is also the sledgehammer Proposition 4.1.7 which might be a fun extra credit homework problem, but which I've almost never found an occassion to actually use beyond the easier table entries beneath it. It follows by systematically extending the method of (6).

If you have occassion to use one of the table entries and you haven't checked it before, you might want to verify it first. Some of these are easy pretty easy:

7) If $f(z)$ has a pole of order k at z_0, and has the form

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

then

$$Res\left(f, z_0\right) = \frac{g^{(k-1)}\left(z_0\right)}{(k-1)!}$$

10) If $f(z)$ has a pole of order k at z_0, with

$$f(z) = \frac{g(z)}{h(z)}$$

where g has a zero of order l and $h(z)$ has a zero of order $k + l$, then for the function

$$\phi(z) = (z - z_0)^k f(z),$$

which has a removable singularity at z_0,

$$Res\left(f, z_0\right) = \lim_{z \to z_0} \frac{\phi^{(k-1)}(z)}{(k-1)!}$$

Residue Theorem (Green's Theorem version): Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \ldots z_k\} \subseteq A$ Let γ be a simple closed contour in A which contains none of the singularities, and which bounds a a subregion B containing some of the singularities. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{z_j \in B} Res\left(f, z_j\right).$$

proof: Use Cauchy's Theorem for domains with holes, Laurent series, and the diagram below.

Exercise: Show - just for the practice - that this theorem includes as special cases

(a) Green's Theorem version of Cauchy's theorem ($\int_\gamma f(z)\, dz = 0$ if f is analytic in A.)

(b) CIF $f\left(z_0\right) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz$ if z_0 is inside γ and f is analytic in A.

Pretty much all of our previous examples of contour integration can be computed via the Residue Theorem, and it's usually the quickest way. New examples become more straightforward.

1) Compute

$$\int_{|\zeta| = 2} \frac{1}{\zeta^2 - 1} \, d\zeta$$

2) Compute

$$\int_{|\zeta| = 2} \frac{e^{\zeta}}{\zeta^2 - 1} \, d\zeta$$

Residue Theorem (Deformation Theorem version, more general than the Green's Theorem version.). Let Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, ... z_k\} \subseteq A$. Let γ be a closed curve which is homotopic to a point in A. Then

$$\int_\gamma f(z)\, dz = 2\,\pi\, i \sum_{j=1}^{k} Res\left(f, z_j\right)\, I\left(\gamma; z_j\right)$$

Being more general than the Green's Theorem version, this proof is also a bit more complicated. For each isolated singularity z_j we have the Laurent series

$$S_j(z) = S_{j1}(z) + S_{j2}(z) = \sum_{n=0}^{\infty} a_{jn}\left(z - z_j\right)^n + \sum_{m=1}^{\infty} \frac{b_{jm}}{\left(z - z_j\right)^m}.$$

Because the z_j are point singularities, the singular part of the series, $S_{j2}(z)$ converges in $\mathbb{C} \setminus \{z_j\}$, and the non-singular part converges for $0 \le |z - z_j| < R_j$ for some positive radius of convergence R_j.

Now consider

$$g(z) := f(z) - \sum_{j=1}^{k} S_{j2}(z).$$

Explain why $g(z)$ has removable singularities at each z_l:

Thus we may consider g to be analytic in A, so since γ is homotopic as closed curves to a point in A,

$$\int_\gamma g(z)\, dz = 0.$$

Expand this to get the result!

Math 4200-001
Week 11 concepts and homework
4.1-4.2
Due Wednesday November 13 at start of class.
Exam will cover thru 4.2

4.1   1de, 3, 5, 7, 9
4.2   2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 8, 9, 13, 15.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)