Math 4200
Monday November 4
3.3 Laurent series: classification of isolated singularities and multiplying Laurent series.
4.1 Calculating residues at isolated singularities

<u>Announcements:</u>

Homework questions?

Isolated singularities table.

Let $f$ be analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$.

| type of singularity at $z_0$ | Laurent series definition | characterization in terms of $\lim_{z\to z_0} f(z)$ |
|---|---|---|
| removable (because $f$ extends to be analytic at $z_0$) | $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in L.S.) | any of: ① $\lim_{z\to z_0} f(z) = L \in \mathbb{C}$ exists ② $\lvert f(z)\rvert \le M$ $\forall\, 0<\lvert z-z_0\rvert \le \rho$ for some $0<\rho<r$. ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$. |
| pole (North pole!) of order N. simple pole if $N=1$ | $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{N} \frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$ | ① $\lim_{z\to z_0} f(z)=\infty$ (the north pole on the Riemann sphere) or ② $\exists\, N$ s.t. $g(z)=(z-z_0)^N f(z)$ has a removable singularity @ $z=z_0$, with $g(z_0)\neq 0$. |
| essential singularity | $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$ with $\{m_j\}\to\infty$, $b_{m_j}\neq 0$ | (Monday) $\forall\, 0<\rho<r$ $\overline{f(D(z_0;\rho)\setminus\{z_0\})} = \mathbb{C}$ ! (In fact, more is true and is called "Picard's Theorem: $f(D(z_0;\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$ except for at most a single point!) e.g. $f(z)=e^{1/z}$ @ $z_0=0$ $f(D(0;\rho)\setminus\{0\})=\mathbb{C}\setminus\{1\}$ $\forall \rho>0$ |

*proof that if f has an essential singularity at* $z_0$, then for each deleted disk $D(z_0;\rho) \setminus \{z_0\}$ in its domain (with arbitrarily small $\rho > 0$), the closure of the image

$$f\left(D(z_0;\rho) \setminus \{z_0\}\right)$$

is *all* of $\mathbb{C}$!

We'll show that if the range statement above is not true, then the singularity at $z_0$ is either removable or a pole, in terms of the Laurent characterizations. Thus the range statement must be true whenever $z_0$ is an essential singularity. Logic!

So, suppose the range statement is false. In other words, there exists some positive $\rho$ so that the closure of $f\left(D(z_0;\rho) \setminus \{z_0\}\right)$ does *not* contain some $w_0 \in \mathbb{C}$. So, since the complement of the closure of the range is open, there is an $\varepsilon > 0$ so that

$$f\left(D(z_0;\rho) \setminus \{z_0\}\right) \cap D(w_0;\varepsilon) = \varnothing .$$

Now, on the domain $D(z_0;\rho) \setminus \{z_0\}$, consider the composition

$$k(z) := \frac{1}{f(z) - w_0} .$$

Show that $k$ has a removable singularity at $z_0$ and solve for $f$ to show that $f$ has either a pole or removable singularity at $z_0$.

multiplying Laurent series term by term is legal:

We already know that we get the coefficient of $(z - z_0)^n$ in the Taylor series of a product $f(z)g(z)$ of analytic functions, by collecting the finite number of terms in the product of the Taylor series for $f$ and $g$ at $z_0$ which have that total power. The analogous statement is true for Laurent series, except that you may be collecting infinitely many terms. (You have a homework problem like this for Wednesday, 3.3.6.)

<u>Theorem</u> Let $f(z), g(z)$ have Laurent series in $A := \{z \mid R_1 < |z - z_0| < R_2\}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} b_m (z - z_0)^{-m} := \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$g(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k .$$

Then $f(z)g(z)$ has Laurent series

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} d_n (z - z_0)^n$$

where

$$d_n = \lim_{N \to \infty} \sum_{j=-N}^{N} a_j \, c_{n-j} .$$

*proof:* Recall from Friday that we can recover the Laurent coefficients for an analytic function with a contour integral. Specifically, if $\gamma$ is any p.w. $C^1$ contour in $A$, with $I(\gamma, z_0) = 1$, e.g. any circle of radius $r$, with $R_1 < r < R_2$, then Laurent coefficients for $f$ are given by

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) (\zeta - z_0)^{m-1} d\zeta,$$

which, if we rewrite the Laurent series by combining the two sums as above,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) (\zeta - z_0)^{-n-1} d\zeta, \quad n \in \mathbb{Z}$$

Thus, fixing $n$, the $n^{th}$ Laurent coefficient $d_n$ for $f(z)g(z)$ is given by

$$d_n = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) g(\zeta) (\zeta - z_0)^{-n-1} d\zeta, \quad n \in \mathbb{Z}.$$

Consider the truncated Laurent series for $f, g$,

$$f_N(z) := \sum_{j=-N}^{N} a_j (z - z_0)^j \ , \ g_N(z) = \sum_{k=-N+n}^{N+n} c_k (z - z_0)^k$$

which converge uniformly to $f, g$ on the contour $\gamma$ as $N \to \infty$, so

$$d_n = \lim_{N \to \infty} \frac{1}{2\pi i} \int_\gamma f_N(\zeta) g_N(\zeta) (\zeta - z_0)^{-n-1} d\zeta,$$

$$= \lim_{N \to \infty} \sum_{j,\, k=-N}^{N} \int_\gamma a_j (\zeta - z_0)^j c_k (\zeta - z_0)^k (\zeta - z_0)^{-n-1} d\zeta$$

by multiplication and term-by-term integration of the finite-sum truncated Laurent series. And picking off the non-zero integrals yields

$$d_n = \lim_{N \to \infty} \sum_{j=-N}^{N} a_j c_{n-j}.$$

Example (relates to hw problem 3.3.6):

a) The function $f(z) = \dfrac{e^{\frac{1}{z}}}{1 - z}$ is analytic for $0 < |z| < 1$. Find its residue at $z_0 = 0$.

b) Let $\gamma$ be the circle of radiius $\frac{1}{2}$ centered at the origin. Find

$$\int_{\gamma} f(z)\, dz.$$

<u>Computing residues</u> In Chapter 4 we'll see lots of examples where we need to compute residues at isolated singularities $z_0$, because they are related to interesting contour integrals or to improper integrals on the real line. In fact section 4.1 is all about residue computation shortcuts in case the situation is complicated. (The residue computations in the homework for this Wednesday are mostly somewhat straightforward.)

In most cases the function of concern is a quotient and the reason for the singularity is that the denominator function has a zero at $z_0$. There's a scary-looking table on page 250 of our text - that will be provided on the next midterm - although I could ask you to verify certain entries in addition to using them. Here are two table entries, one easy, one a bit more complicated. We have everything we need to check these table entries, because we know how to multiply Taylor series and Laurent series:

1) Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$. Prove that $f$ has a pole of order $1$, and

$$Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

2) Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0, h(z_0) = h'(z_0) = 0, h''(z_0) \neq 0$. Then $f$ has a pole of order $2$ and

$$Res(f, z_0) = \frac{2\, g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{h''(z_0)^2} \qquad !!!$$