Notice that

$$f(z) = \frac{z-a}{z-b}\left(\frac{c-b}{c-a}\right)$$

maps

$$\begin{aligned} a &\to 0 \\ b &\to \infty \\ c &\to 1. \end{aligned}$$

Since 3 points uniquely determine particular circles one can use FLT's to map any circle or line to any other circle or line.

Using functions of this form, and their inverses, one can construct FLT's to map triples of points to triples of points:

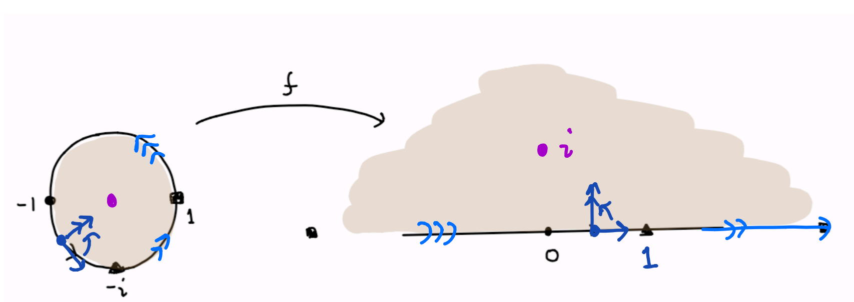
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Thus you can map any line or circle to any other line or circle.

<u>Example</u> Find a FLT from the unit disk to the upper half plane by mapping

$$\left.\begin{aligned} -1 &\to 0 \\ 1 &\to \infty \\ -i &\to 1 \end{aligned}\right\}$$

and making any necessary adjustments. (By magic, once you know the boundary of the disk goes to the real axis, you only have to check that one interior point goes to an interior point, or that the orientation is correct along the boundary, to know that you're mapping the unit disk to the upper half plane instead of the lower half plane. We'll prove a general magic theorem along these lines on Wednesday.)



$$f(z) = \frac{z+1}{z-1}\underbrace{\frac{-i-1}{-i+1}\,\frac{i+1}{i+1}}$$

$$\frac{1-i-i-1}{2} = \frac{-2i}{2} = -i$$

$$\boxed{f(z) = -i\left(\frac{z+1}{z-1}\right)}$$

$$f(0) = -i\left(\frac{1}{-1}\right) = i$$

Example Find a conformal transformation of the first quadrant to the unit disk, so that the image of $1+i$ is the origin. How many such conformal transformations are there? It's fine to write your transformation as a composition.

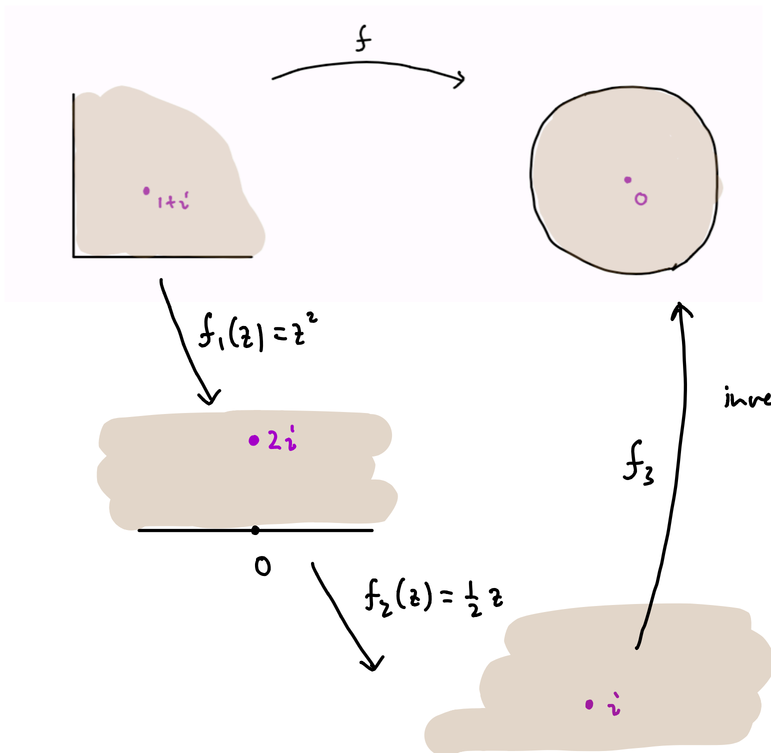
$$\text{old } f = \frac{-iz-i}{z-1}$$

$$[f] = \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}$$

$$[f]^{-1} = \cancel{\tfrac{1}{2i}} \begin{bmatrix} -1 & i \\ -1 & -i \end{bmatrix}$$

$$f_3(z) = \frac{-z+i}{-z-i} = \frac{z-i}{z+i}$$

$i \mapsto 0$



inverse fn from previous page

$$f = e^{i\theta} f_3 \circ f_2 \circ f_1$$

Math 4200
Wednesday November 27
5.2 conformal maps and fractional linear transformations

Announcements: We'll begin by finishing Monday's examples

M : Bjorn, Alexandrea, Keegan : Hyperbolic plane/dish
W : Sage, Jackson : Riemann-Zeta fcn and prime numbers
F : Chase, Preston : Mandelbrott set & Julia set fractals
Josh : quaternions

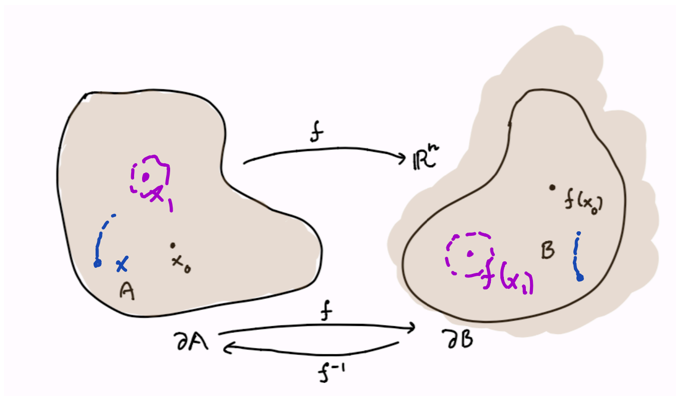
optional, since it's reading day. I'll reserve a room for 1.5 hours

§5.2 HW at end of today's notes

Magic Theorem Let $A, B \subseteq \mathbb{R}^n$ be open, connected, bounded sets.
Let $f: A \to \mathbb{R}^n$, $f \in C^1$, with $df_x : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ invertible $\forall\ x \in A$ (i.e. the Jacobian matrix is invertible). Furthermore, assume

- $f: \overline{A} \to \mathbb{R}^n$ is continuous and one-to-one.
- $f(\delta A) = \delta B$
- $f(x_0) \in B$ for at least one $x_0 \in A$.

Then $f(A) = B$ and f is a global *diffeomorphism* between A and B. (i.e. $f^{-1} : B \to A$ is also differentiable), and $f^{-1}: \overline{B} \to \overline{A}$ is continuous.



proof: Step 1: $f(A) \subseteq B$.
proof: Let

$$\mathsf{O} := \{x \in A \mid f(x) \in B\}$$

Then

- $x_0 \in \mathsf{O}$
- O is open by the local inverse function theorem, since $x_1 \in \mathsf{O}$ and $f(x_1) \in B$ implies there is a local inverse function from an open neighborhood of $f(x_1)$ in B, back to a neighborhood of x_1 in A.
- O is closed in A because if $\{x_k\} \subseteq \mathsf{O}$, $\{x_k\} \to x \in A$ then $\{f(x_k)\} \to f(x)$ and since $\{f(x_k)\} \subseteq B$ we have $f(x) \in \overline{B}$. But since f is one-one and maps the boundary of A bijectively to the boundary of B, $f(x)$ cannot be in the boundary of B. Thus $f(x) \in B$.
- Thus, since A is connected, O is all of A, and $f(A) \subseteq B$.

Step 2: $f(A) = B$.
proof:

- $f(A)$ is open (by the local inverse function theorem again), so $f(A) \subseteq B$ is open.
- And $f(A)$ is closed in B because if

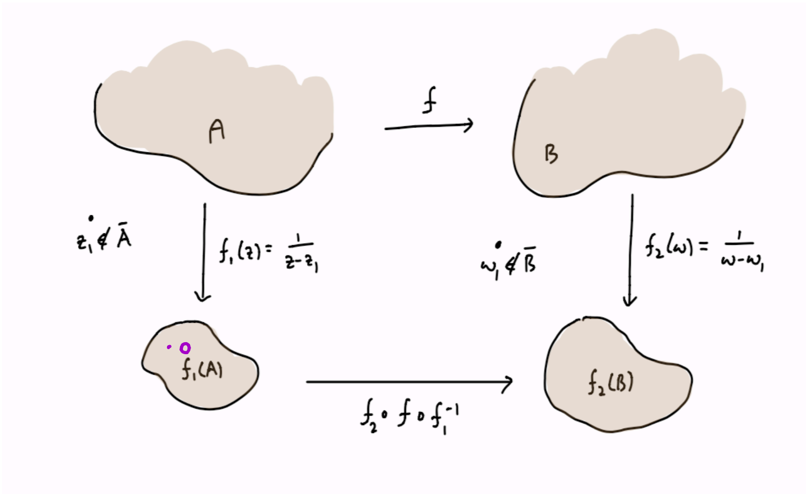
$$\{f(x_k)\} = \{y_k\} \subseteq f(A), \text{ with } \{y_k\} \to y \in B,$$

then because $\overline{A}$ is compact, a subsequence $\{x_{k_j}\} \to x \in \overline{A}$ with $\{f(x_{k_j})\} \to f(x) = y$, so $x \notin \delta A$ because $y \in B$, so $x \in A$ and $y \in f(A)$.

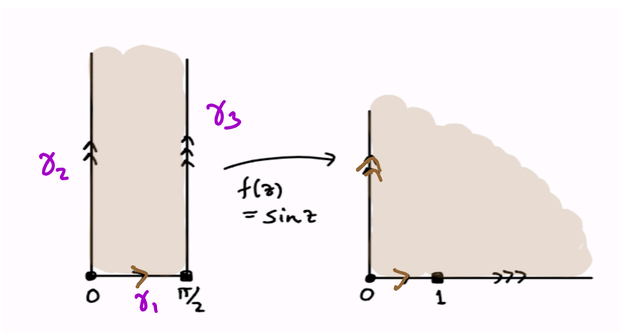
- So, because B is connected, $f(A)$ is all of B.

QED.

Remark: In $\mathbb{C}$ you can also imply this theorem to unbounded domains, i.e. in $\mathbb{C} \cup \{\infty\}$ because of the following diagram, in which $f_2 \circ f \circ f_1^{-1}$ satisfies the hypotheses of the original theorem:



Example: Show that $f(z) = \sin(z)$ is a conformal equivalence from the indicated open half strip to the open first quadrant. The savings is that by checking the boundary map and that at least one interior point gets mapped appropriately, and that the map is $1-1$ we get the "onto" for free.



$\cosh y = \frac{1}{2}(e^y + e^{-y})$
$\sinh y = \frac{1}{2}(e^y - e^{-y})$

- verify the boundary maps $1-1$ onto the boundary. It helps to use

$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$
$= \sin x \cosh y + i \cos x \sinh y$

on: γ_1: $\sin(z) = \sin x \cdot 1$
$0 \le x \le \pi/2 \Rightarrow 0 \le \sin x \le 1$

on γ_2: $\sin(iy) = i \sinh y$
$0 \le y \le \infty$, $0 \le \sinh y \le \infty$

on γ_3: $\sin(\frac{\pi}{2} + iy)$
$= \cosh y$
$0 \le y \le \infty$, $1 \le \cosh y \le \infty$

- Verify $1-1$

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

(x_1, y_1)
(x_2, y_2) in A: $\underline{x_1 \le x_2}$ If $\sin x_1 \cosh y_1 + i \cos x_1 \sinh(y_1)$
$= \sin x_2 \cosh y_2 + i \cos x_2 \sinh(y_2)$

- $\frac{d}{dz}(\sin(z)) = \cos(z)$ is non-zero in A.

Re: $x_1 \le x_2 \Rightarrow \sin x_1 \le \sin x_2$
$\Rightarrow \cosh y_1 \ge \cosh y_2$
$y_1 \ge y_2$

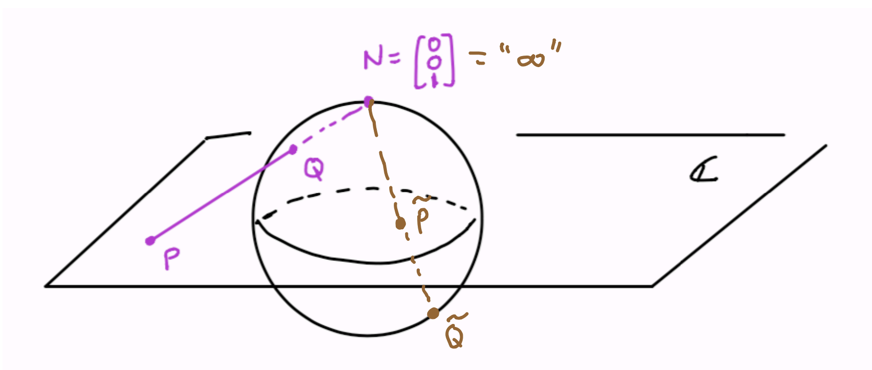
Im: $x_1 \le x_2 \Rightarrow \cos x_1 \ge \cos x_2 \Rightarrow \sinh y_1 \le \sinh y_2$
$y_1 \le y_2$

So $y_1 = y_2 \Rightarrow x_1 = x_2$ Whew!

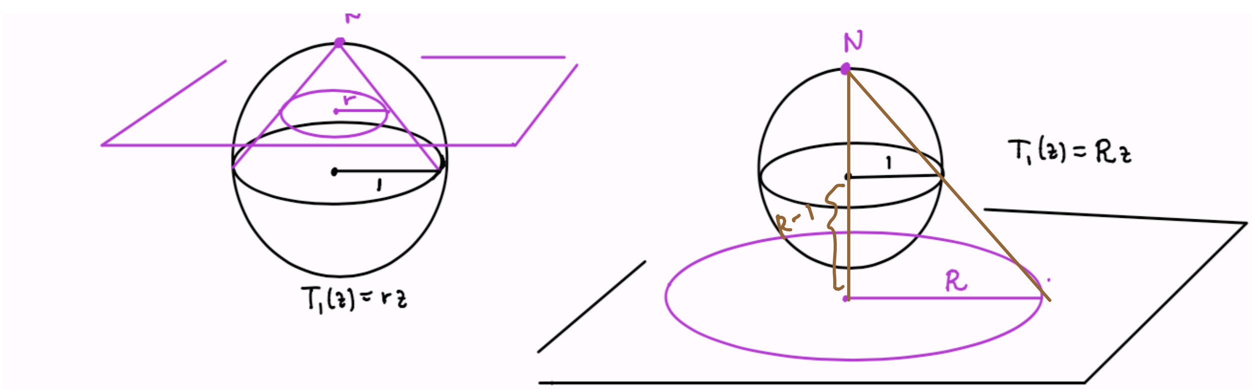
Fractional linear transformations turn out to be precisely the set of all conformal differmorphisms of the Riemann sphere. (Analytic functions on the Riemann sphere are meromorphic functions with a finite number of isolated singularities (which are all poles), and such that $f\left(\frac{1}{z}\right)$ has the same property. The requirement that they're 1-1 implies that they're FLT's.) And since the Riemann sphere can be identified with the unit sphere in $\mathbb{R}^3$ via stereographic projection, it makes sense that fractional linear transformations can be understood in that context. In a way, what's going on in that context is easier to understand!

Identify the Riemann sphere with the unit sphere in $\mathbb{R}^3$ via stereographic projection from the north pole $(0, 0, 1) \in \mathbb{R}^3$: One can check that stereographic projection and its inverse are actually conformal (angle preserving), and that stereographic projection maps circles on the unit sphere to circles and lines in $\mathbb{R}^2 = \mathbb{C}$, and vise verse. Fractional linear transformations in $\mathbb{C}$ correspond via various stereographic projections, to various Euclidan motions of the unit sphere in $\mathbb{R}^3$!

$St(Q) = P \in \mathbb{C}$, $St^{-1}(P) = Q$: In the initial case we use the unit sphere centered at the origin, and we always project from the north pole:



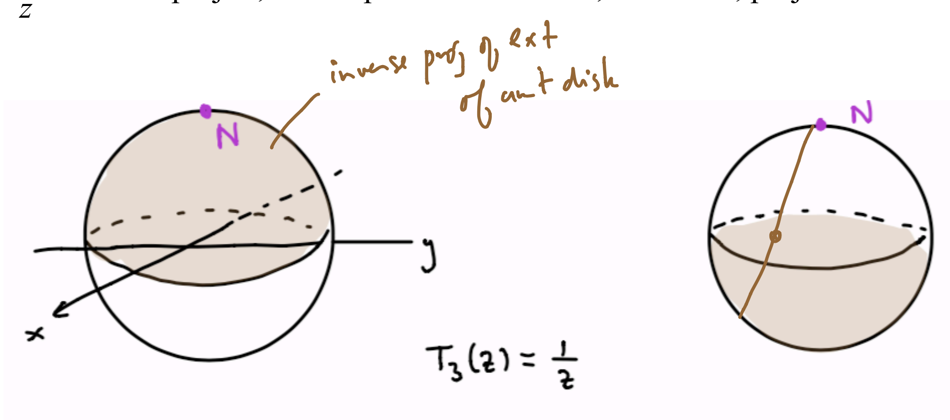
dilations: inverse project, then lift or lower the sphere before projectiong back onto the $x - y$ plane. The picture is tracking what's happening to the original unit circle in $\mathbb{C}$.



rotations in $\mathbb{C}$ inverse project, then rotate the sphere about the vertical axis, then project.

translations in $\mathbb{C}$: inverse projection, translate the sphere horizontally, project.

<u>inversions</u> $z \to \frac{1}{z}$: inverse project, rotate sphere about x axis, π radians, project.



<u>How could you get LFT's from the unit disk to the upper half plane?</u>

There's a movie!

https://www.youtube.com/watch?v=JX3VmDgiFnY

Math 4200-001

Week 14 concepts and homework

5.2

Due Wednesday December 4, but accepted until Friday December 6 at 5:00 p.m.

5.2   1, 4a, 6, 7, 9, 10, 17, 24, 26, 33, 34