Math 4200-001

Week 13 concepts and homework

4.4

Due Wednesday November 27 at start of class.

4.4: 2, 3, 4, 5, 8, 9

5.1: 10, 11, 12.

Math 4200
Monday November 25
Chapter 5: 5.1-5.2 conformal maps and fractional linear transformations

Announcements:

HW 5.1.12 : Every conformal bijection
$f: \mathbb{C} \to \mathbb{C}$
is given by $f(z) = az + b$!

hint : $\nexists$ $f: \mathbb{C} \to \mathbb{C}$ which is one to one
(let alone onto),
except $f(z) = az + b$, $a \neq 0$.

If f exists,
it's entire, i.e.
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbb{C}.$$
show that actually $a_n = 0 \ \forall n \geq 2$
play with ideas from § 3.3

<u>Riemann Mapping Theorem</u> (version 2)

Let $A, B \subseteq \mathbb{C}$ be open and simply connected but not all of $\mathbb{C}$.

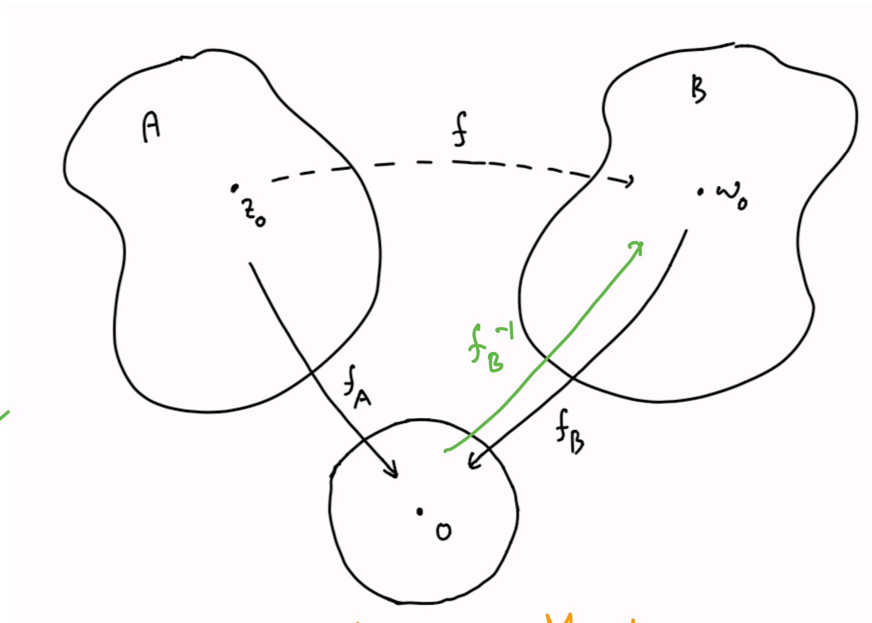
Let $z_0 \in A$, $w_0 \in B$.

Then $\exists! f: A \to B$ such that f is a conformal bijection satisfying

$$f(z_0) = w_0$$
$$f'(z_0) \text{ is real and positive.}$$

proof: Chase the diagram arrows below to prove existence, and then uniqueness, letting f_A, f_B be as in version 1 on the previous page.

$\exists$, !



$\exists$: $f_B^{-1} \circ f_A$

try this f.

$f_B^{-1}(f_A(z_0)) = w_0$. ✓

$f^{-1}: B \to A$

$f^{-1} = f_A^{-1} \circ f_B$ ✓

$f'(z_0) \in \mathbb{R}^+$?

$\parallel$

$\underbrace{(f_B^{-1})'(0)}_{\frac{1}{f_B'(w_0)}} f_A'(z_0)$

$\frac{1}{f_B'(w_0)} f_A'(z_0) = (+)(+) > 0$. ✓

<u>uniqueness on Monday</u>

If some f exists as per Theorem

$\Rightarrow g := f_B \circ f \circ f_A^{-1}$ $\qquad g: D(0;1) \to D(0;1)$

$0 \mapsto z_0 \mapsto w_0 \mapsto 0$.

$g(0) = 0$.

chain rule $\quad g'(0) = f_B'(w_0) f'(z_0) \underbrace{(f_A^{-1})'(0)}_{\frac{1}{f_A'(z_0)}}$

$= (+)(+)(+) > 0$

recall, the only $g: D(0;1) \to D(0;1)$ with $g(0) = 0$, $g'(0) > 0$ is $g(z) = z$

$\Rightarrow f_B \circ f \circ f_A^{-1} = id$

$\Rightarrow f \circ f_A^{-1} = f_B^{-1} \Rightarrow f = f_B^{-1} \circ f_A$ ∎

The maps we were missing in some of the Friday examples were compositions of the ones we found, with *fractional linear transformations*, of which Mobius transformations of the unit disk are examples.

Def a *fractional linear transformation (FLT)* $f: \mathbb{C} \cap \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function defined by

$$f(z) = \frac{a\,z + b}{c\,z + d},$$

where $a, b, c, d \in \mathbb{C}$ and

$$ad - bc = det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 .$$

Note that when the determinant does equal zero, the numerator and denominator of f are multiples so the function f is just a constant. And otherwise $f'(z)$ is never zero. Also, one could normalize the determinant to be $+1$ by dividing all of the coefficients by the same number (a square root of the determinant).

compute $f'(z)$ to verify the claims above.

$$f'(z) = \frac{a(\cancel{cz} + d) - (\cancel{az} + b)c}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2}$$

Example $f(z) = a\,z + b = \dfrac{a\,z + b}{0\,z + 1}$. You will show in your homework that these are the only one-to-one conformal maps defined on all of $\mathbb{C}$. Notice that they are conformal bijections of $\mathbb{C}$. A good place to start in your homework problem is with the Taylor series for the entire function f, which converges on all of $\mathbb{C}$, and see what happens if you require f to be $1 - 1$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Related Exercise In the Riemann Mapping Theorem we assumed our open, simply connected domain was not $\mathbb{C}$. Why is there no conformal bijection $f: \mathbb{C} \to \mathrm{D}(0; 1)$? It's a one-line answer if you can think of it.

<u>The algebra of fractional linear transformations</u>: Show that if

$$f(z) = \frac{az+b}{cz+d}$$

$$g(w) = \frac{\alpha w+\beta}{\gamma w+\delta}$$

$$g(f(z)) = \frac{\alpha\left(\frac{az+b}{cz+d}\right)+\beta}{\gamma\left(\frac{az+b}{cz+d}\right)+\delta}$$

$$= \frac{\alpha(az+b)+\beta(cz+d)}{\gamma(az+b)+\delta(cz+d)}$$

$$= \frac{(\alpha a+\beta c)z+\alpha b+\beta d}{(\gamma a+\delta c)z+\gamma b+\delta d}$$

Then

$$g(f(z)) = \frac{Az+B}{Cz+D}$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

A matrix for the composition FLT can be obtained by multiplying the matrices for the individual FLTs!

Geometers would say: The *group* $SL(2,\mathbb{C})$ ("The *special linear group* of 2×2 matrices with entries from $\mathbb{C}$) *acts* on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The word *special* refers to the fact that each FLT can be represented uniquely with a matrix having determinant exactly equal to 1, and the word *acts* refers to the fact that matrix multiplication (the group operation) in the group, corresponds to composition of transformations on the Riemann sphere.

collection of objects
with a multiplication (closed) &
with multiplicative inverses
& an identity.

these matrices
are association
with transformations
of a space ($\mathbb{C} \cup \{\infty\}$).
for us.

Example Let

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

$$[f] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a (non-constant) FLT. Use matrix algebra to find a formula for $f^{-1}(z)$.

$$[f^{-1}] = \cancel{\frac{1}{ad-bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$f^{-1}(w) = \frac{dw - b}{-cw + a}.$$

$$f(z) = \frac{az + b}{cz + d}.$$

Corollary Fractional linear transformations are bijections of the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

for $z \in \mathbb{C}$, $w = f(z) \in \mathbb{C}$, you have an inverse fcn.

check ∞'s separately:

$$\infty \xrightarrow[f]{} \frac{a}{c} \xrightarrow[f^{-1}]{} \infty$$

$(c \neq 0)$

$$-\frac{d}{c} \xrightarrow[f]{} \infty \xrightarrow[f^{-1}]{} -\frac{d}{c} \quad \checkmark$$

Theorem Fractional linear transformations map the set of all circles and lines to itself.

proof: Any circle or line in the $x - y$ plane can be described implicitly as the solution set to an equation

$$(1) \qquad A(x^2 + y^2) + Bx + Cy + D = 0$$

where $A, B, C, D \in \mathbb{R}$ and ~~are~~ A, B, C not all zero.

First, show that

$$T_1(z) = z + a \quad \text{(translation)} \qquad \text{preserve } \{\text{circles, lines}\}$$
$$T_2(z) = cz \quad \text{(rotation-dilation)} \qquad \text{"} \qquad \text{"}$$
$$T_3(z) = \frac{1}{z} \quad \text{(inversion)} \qquad ??$$

convert the solution set of an equation of form (1) into the solutions set of a (different) equation of form (1).

Let $A(x^2+y^2) + Bx + Cy + D = 0$

$z = x + iy$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u + iv$$

$$A + \frac{Bx}{x^2+y^2} + \frac{Cy}{x^2+y^2} + \frac{D}{x^2+y^2} = 0$$

$$u^2 + v^2 = \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$$

$$A + Bu - Cv + D(u^2+v^2) = 0$$

Then show that any fractional linear transformation

$$f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d)}{cz+d} + \frac{b - \frac{ad}{c}}{cz+d}$$

$\frac{a}{c}$

is a composition of translations, rotation-dilations, and inversions. Hint: Treat $c = 0$, $c \neq 0$ separately. If $c \neq 0$ first do something equivalent to long division to rewrite f.

$f_1 : \quad z \longmapsto cz = z_1 \qquad cz$

$f_2 : \quad z_1 \longmapsto z_1 + d = z_2 \qquad cz + d$

$f_3 : \quad z_2 \longmapsto \frac{1}{z_2} = z_3 \qquad \frac{1}{cz+d}$

$f_4 : \quad z_3 \longmapsto \left(b - \frac{ad}{c}\right) z_3 = z_4 \qquad \frac{b - \frac{ad}{c}}{cz+d}.$

$f_5 : \quad z_4 \longmapsto \frac{a}{c} + z_4.$

$$f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1 = f$$

We've seen the circle-line property illustrated with pictures for the Möbius transformations

Remark: We actually proved both existence and uniqueness for this version the RMP earlier in the course, using Mobius transformations, for the very special case of

$$f : \mathrm{D}(0;1) \to \mathrm{D}(0;1)$$
$$f(z_0) = 0$$
$$f'(z_0) \text{ is real and positive.}$$

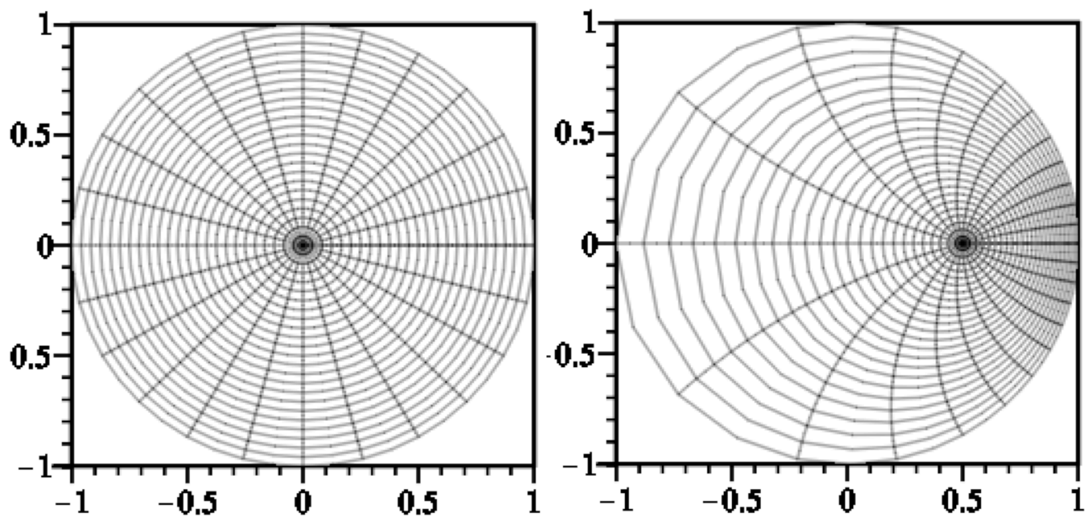
The mapping is

$$f(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

$$f'(z) = \frac{1 \cdot (1 - \bar{z}_0 z) - (z - z_0)(-\bar{z}_0)}{(1 - \bar{z}_0 z)^2}$$

$$@\ z_0 : \ \frac{1 - |z_0|^2}{(1 - |z_0|^2)^2} > 0.$$

$$f(z) = \frac{z - .5}{1 - .5z}$$



Notice that

$$f(z) = \frac{z-a}{z-b}\left(\frac{c-b}{c-a}\right)$$

maps

$$a \to 0$$
$$b \to \infty$$
$$c \to 1.$$

Since 3 points uniquely determine particular circles one can use FLT's to map any circle or line to any other circle or line.

Using functions of this form, and their inverses, one can construct FLT's to map triples of points to triples of points:

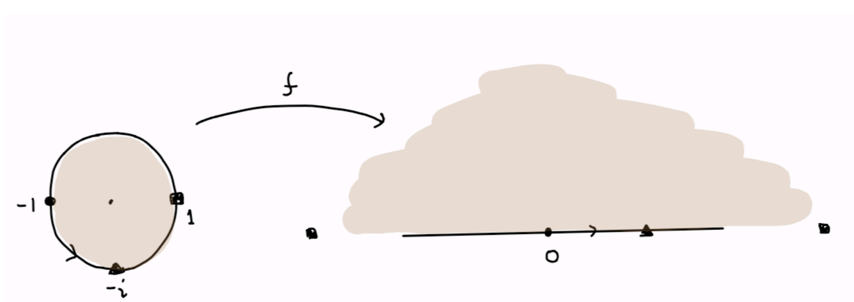
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Thus you can map any line or circle to any other line or circle.

<u>Example</u> Find a FLT from the unit disk to the upper half plane by mapping

$$-1 \to 0$$
$$1 \to \infty$$
$$-i \to 1$$

and making any necessary adjustments. (By magic, once you know the boundary of the disk goes to the real axis, you only have to check that one interior point goes to an interior point, or that the orientation is correct along the boundary, to know that you're mapping the unit disk to the upper half plane instead of the lower half plane. We'll prove a general magic theorem along these lines on Wednesday.)



Example Find a conformal transformation of the first quadrant to the unit disk, so that the image of $1 + i$ is the origin. How many such conformal transformations are there? It's fine to write your transformation as a composition.