Math 4200
Friday November 22
Chapter 5: Conformal maps. This is an in-depth return to ideas we began the course with.

Announcements:

Are there any questions on Wed notes?

Monday after Thanksgiving

Alexandrea
Keegan
Bjorn

$\left.\begin{matrix}\text{Alexandrea}\\ \text{Keegan}\\ \text{Bjorn}\end{matrix}\right\}$ hyperbolic plane

Recall the chain rule for curves:

If $f$ is analytic at $z_0$ and
$\gamma : I \to \mathbb{C}$ a differentiable path, with $\underbrace{\gamma(t_0) = z_0}$,

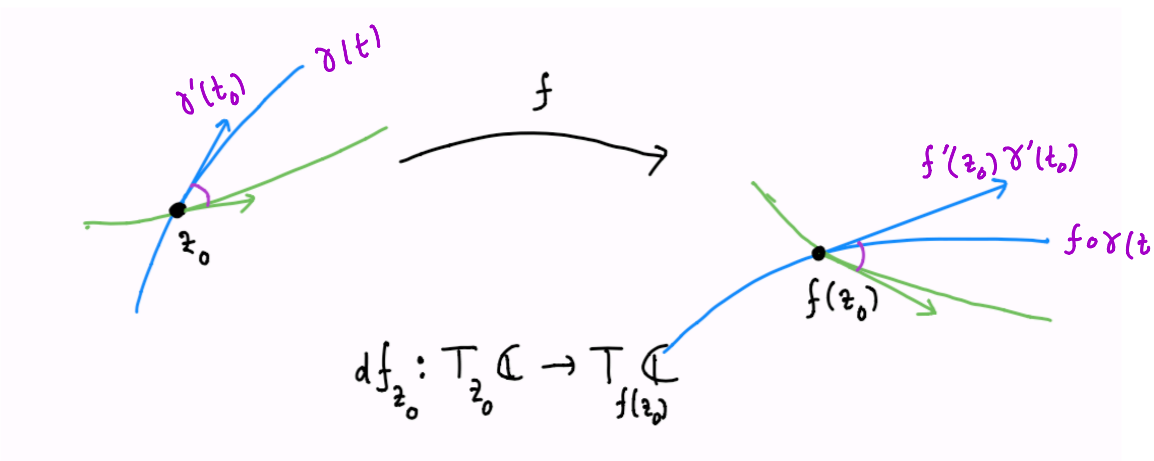
Then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = \underbrace{f'(z_0)}\underbrace{\gamma'(t_0)}.$$

In other words, the differential map $df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$$
$$\gamma'(t_0) \to f'(z_0)\gamma'(t_0)$$

converts tangent vectors based at $z_0$ into ones based at $f(z_0)$, via a rotation amount $arg\, f'(z_0)$ and a scaling by $|f'(z_0)|$.

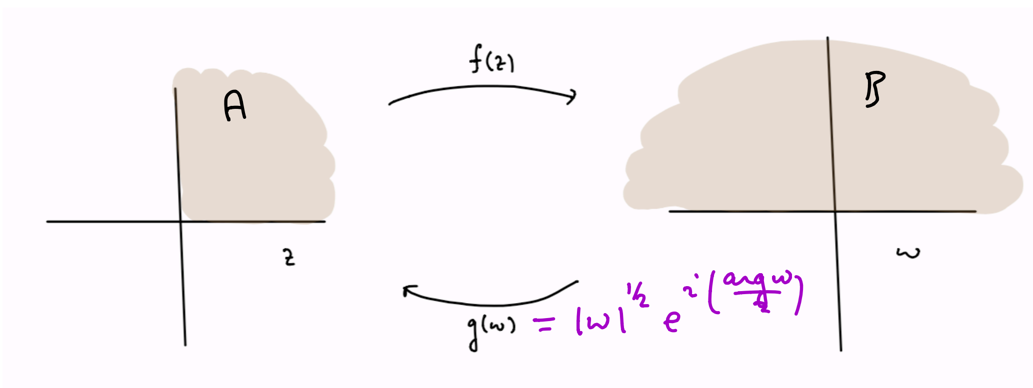



We called a differentiable map $f : A \to \mathbb{C}$ *conformal* on $A$ if this rotation and scaling property holds $\forall\, z \in A$. And we know by the Cauchy-Riemann equations that this rotation-dilation property on tangent vectors is equivalent to $f : A \to \mathbb{C}$ analytic, with $f'(z) \neq 0\ \forall\, z \in A$.

In Chapter 5 we are interested in finding bijective conformal maps between open connected domains $A, B \subseteq \mathbb{C}$. In such cases we will call $A$ and $B$ *conformally equivalent.* Applications include partial differential equations and geometry.

Example: How many conformal bijections can we find between the $1^{st}$ quadrant and the upper half plane? Just to be clear, we're focusing on the open domains and not worrying about boundary behavior at this point.

$f(z) = z^2$, $g(w) = \sqrt{w}$ is one pair. What others can we find?




$g(w) = |w|^{1/2} e^{i\left(\frac{\arg w}{2}\right)}$

$0 < \arg w < \pi.$

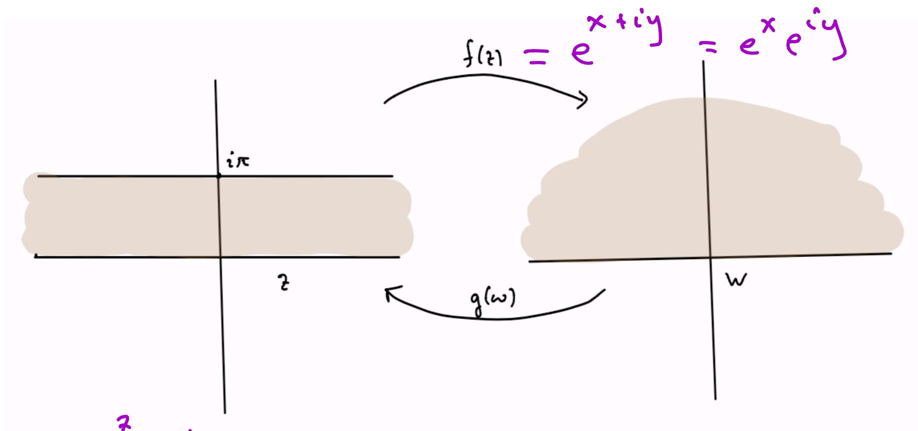
$f(z) = az^2 \qquad a > 0$

$f(z) = az^2 + b \qquad b \in \mathbb{R}$

there's actually more!

Example  Same question for $A$ equal the strip for which $0 < \mathrm{Im}(z) < \pi$, and $B$ equal to the upper half plane.

one pair: $f(z) = e^z$, $g(w) = \log(w)$.



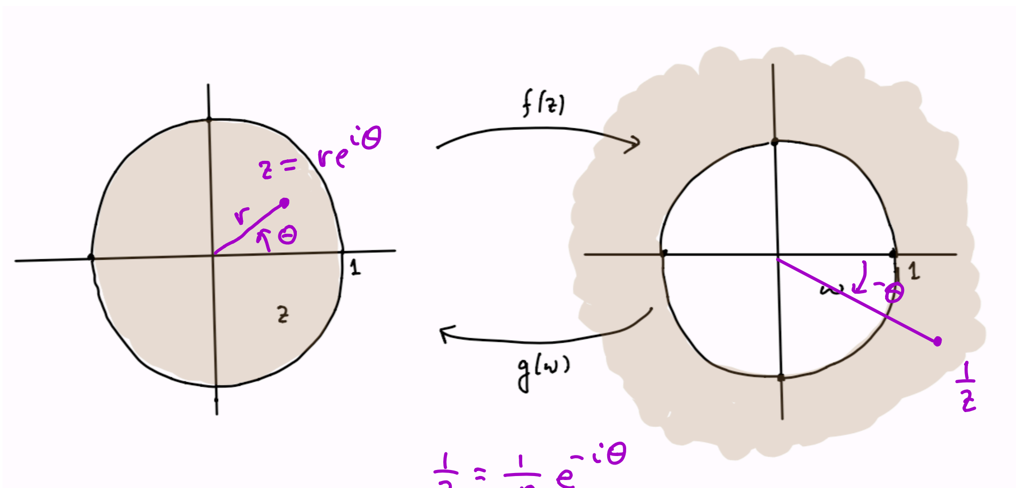

$f(z) = a e^z + b \qquad a > 0,\ b \in \mathbb{R}.$

(more!)

Example Same question for $A$ equal to the open unit disk, $B$ equal to the complement of the closed unit disk.

one pair: $f(z) = \dfrac{1}{z}$, $g(w) = \dfrac{1}{w}$

$f : D(0;1) \to \mathbb{C} \cup \{\infty\}$.




$\frac{1}{z} = \frac{1}{r} e^{-i\theta}$

except @ $z=0$ : pole. $f(0) = \infty$.

$f(z) = e^{i\alpha} \dfrac{1}{z}$

I'm sure you remember (/s).

$z_0 \in D(0;1)$.

$g(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$ Möbius transformations. $g : D(0;1) \to D(0;1)$

$g(z_0) = 0$.

$f(z) = e^{i\alpha} \dfrac{1}{g(z)}$

3 free real parameters!

$z_0$ is 2.
$\alpha$ is 1.

these turn out to be all possible $f$!

Chances are that unless our memory is really good we missed some examples .... see the following two versions of the Riemann Mapping Theorem.

Riemann Mapping Theorem (version 1)
Let $A \subseteq \mathbb{C}$ (but $A \neq \mathbb{C}$) be open and simply connected.
Let $z_0 \in A$.
Then $\exists! f: A \to D(0;1)$ such that $f$ is a conformal bijection satisfying

$$f(z_0) = 0$$
$$f'(z_0) \text{ is real and positive.}$$

Note that there are three real degrees of freedom for conformal bijections with the disk: 2 from the choice of $z_0$ and one from the choice of the argument of $f'(z_0)$ (since we can always compose an $f$ with $f(z_0) = 0$ by a rotation about the origin to uniquely make the argument of $f'(z_0)$ whatever we want).

The proof that such maps exist for any open simply connected subset of $\mathbb{C}$ except $\mathbb{C}$ itself would take several lectures to explain and we won't do it in this course. But we already have the tools to prove uniqueness. In fact, these ideas recall class discussions and homework.

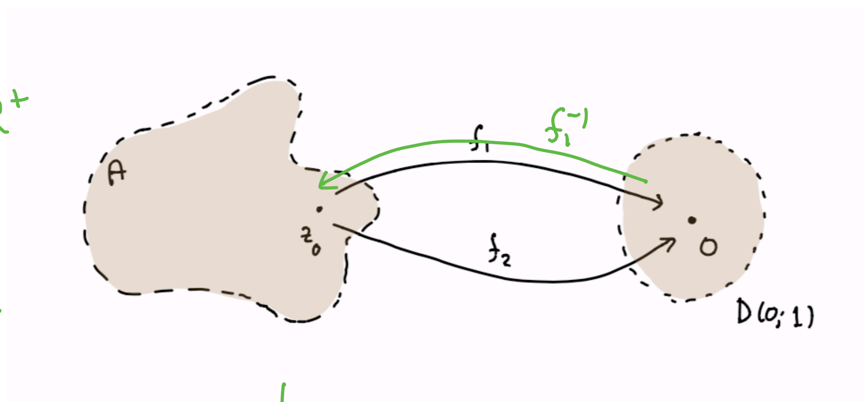
*proof of uniqueness*: Suppose $f_1, f_2$ satisfy the conditions above. Define

$$g := f_2 \circ f_1^{-1} : D(0;1) \to D(0;1).$$

and apply (recall) ideas related to Schwarz's lemma for $g$ and $g^{-1}$ that we applied earlier in the course.

$f_1(z_0) = 0$ $\qquad$ $f_2(z_0) = 0$

$f_1'(z_0) \in \mathbb{R}^+$ $\qquad$ $f_2'(z_0) \in \mathbb{R}^+$




$g := f_2 \circ f_1^{-1} : D \to D.$

$g(0) = f_2(z_0) = 0.$

$g'(0) = f_2'(z_0) \underbrace{(f_1^{-1})'(z_0)}_{\frac{1}{f_1'(0)}} = (+)(+) > 0.$

$f(f^{-1}(z)) = z$

$\underbrace{f'(f^{-1}(z_0))}_{f'(0)} \cdot (f^{-1})'(z_0) = 1$

Consider
$$G(z) = \begin{cases} \frac{g(z)}{z} & z \neq 0 \\ g'(0) & z = 0. \end{cases}$$ is cont. on D.

So sing @ $z=0$ is removable, $G$ is analytic on $D(0;1)$.

Let $0 < r < 1$ Max princ on $D(0;r)$ says $|G(z)| \leq$ max on r-circle $|\frac{g(z)}{z}|$ $\leq \frac{1}{r}$

Let $r \to 1 \Rightarrow |G(z)| \leq 1 \quad \forall z \in D(0;1).$

(1) $\Rightarrow |g(z)| \leq |z| \quad \forall z \in D(0;1).$

Apply the same reasoning to

$g^{-1}(z) = f_1 \circ f_2^{-1}$

$(2) \Rightarrow |g^{-1}(z)| \leq |z| \quad \forall z \in D(0;1)$.

$$|z| = |g(g^{-1}(z))| \underset{(1)}{\leq} |g^{-1}(z)| \underset{(2)}{\leq} |z|$$

$\Rightarrow |g^{-1}(z)| = |z|$

also $|g(z)| = |z|$.

$$\left( |z| = |g^{-1}(g(z))| \underset{(2)}{\leq} |g(z)| \underset{(1)}{\leq} |z| \right)$$

So $|G(z)| \equiv 1$. on $D(0;1)$

so by interior case of max princ.

$G(z) = C_1 = e^{i\alpha}$

$\dfrac{g(z)}{z} = e^{i\alpha}$

$g(z) = e^{i\alpha} z$.

$g'(0) = e^{i\alpha} = 1$

must be pos real.

$g(z) = z$

$f_2 \circ f_1^{-1}(z) = z$

i.e. $f_2 = (f_1^{-1})^{-1} = f_1$

Remark: We actually proved both existence and uniqueness for this version the RMP earlier in the course, using Mobius transformations, for the very special case of

$$f : \mathrm{D}(0; 1) \to \mathrm{D}(0; 1)$$
$$f(z_0) = 0$$
$$f'(0) \text{ is real and positive.}$$

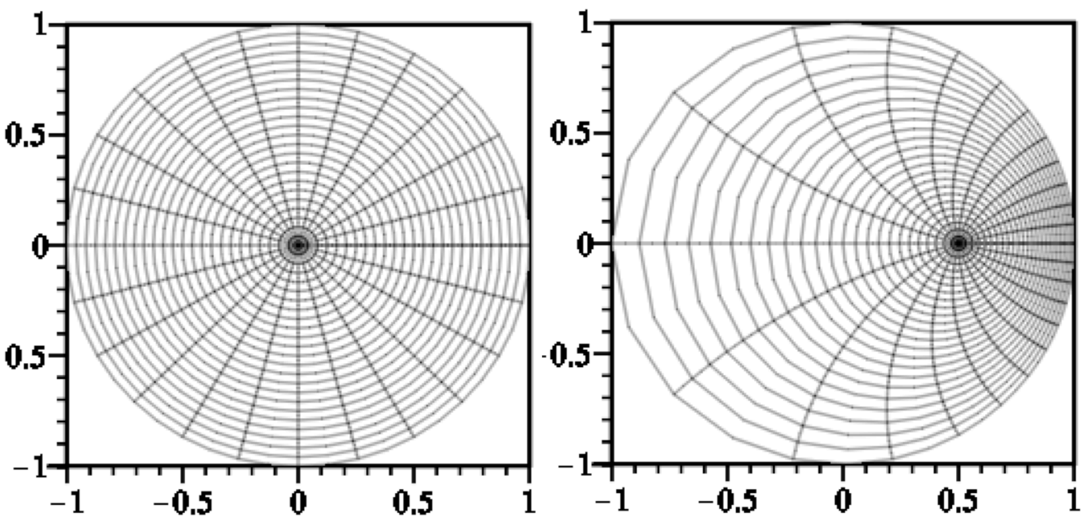
The mapping is

$$f(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

$$f'(z) = \frac{1 \cdot (1 - \bar{z}_0 z) - (z - z_0)(-\bar{z}_0)}{(1 - \bar{z}_0 z)^2}$$

$$@\ z_0: \quad \frac{1 - |z_0|^2}{(1 - |z_0|^2)^2} > 0.$$

$$f(z) = \frac{z - .5}{1 - .5z}$$



Riemann Mapping Theorem (version 2)

Let $A, B \subseteq \mathbb{C}$ be open and simply connected but not all of $\mathbb{C}$.
Let $z_0 \in A$, $w_0 \in B$.
Then $\exists! f: A \to B$ such that $f$ is a conformal bijection satisfying

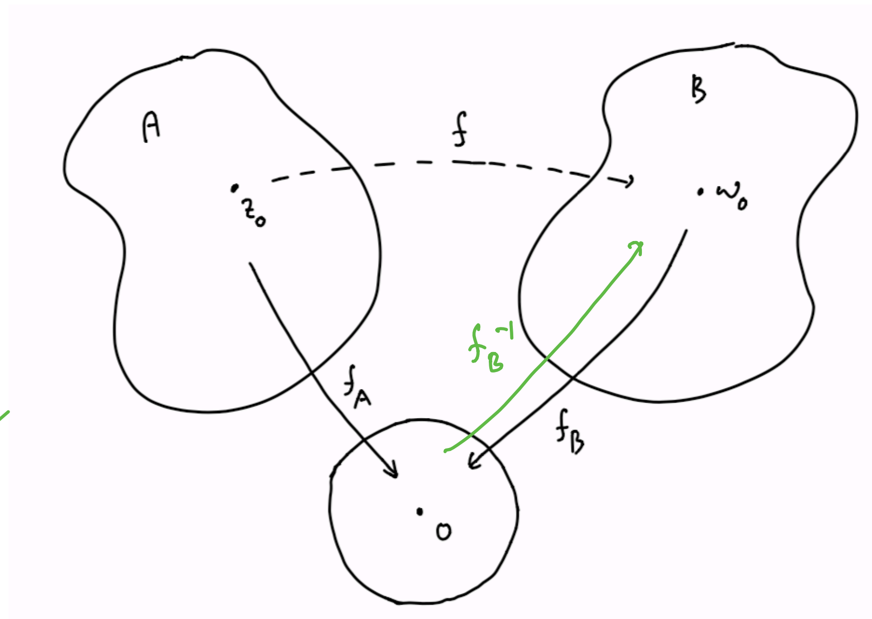
$$f(z_0) = w_0$$
$$f'(z_0) \text{ is real and positive.}$$

*proof:* Chase the diagram arrows below to prove existence, and then uniqueness, letting $f_A, f_B$ be as in version 1 on the previous page.

$\exists$, !

$\exists$: $f_B^{-1} \circ f_A$

try this $f$.




$f_B^{-1}(f_A(z_0)) = w_0$. ✓

$f^{-1}: B \to A$

$f^{-1} = f_A^{-1} \circ f_B$ ✓

$f'(z_0) \in \mathbb{R}^+$ ?

$$f'(z_0) = \underbrace{(f_B^{-1})'(0)}_{\frac{1}{f_B'(w_0)}} f_A'(z_0)$$

$\frac{1}{f_B'(w_0)} f_A'(z_0) = (+)(+) > 0$. ✓