Math 4200
Friday November 22
Chapter 5: Conformal maps. This is an in-depth return to ideas we began the course with.

Announcements:

Recall the chain rule for curves:

If f is analytic at z_0 and
$\gamma : I \to \mathbb{C}$ a differentiable path, with $\gamma(t_0) = z_0$,

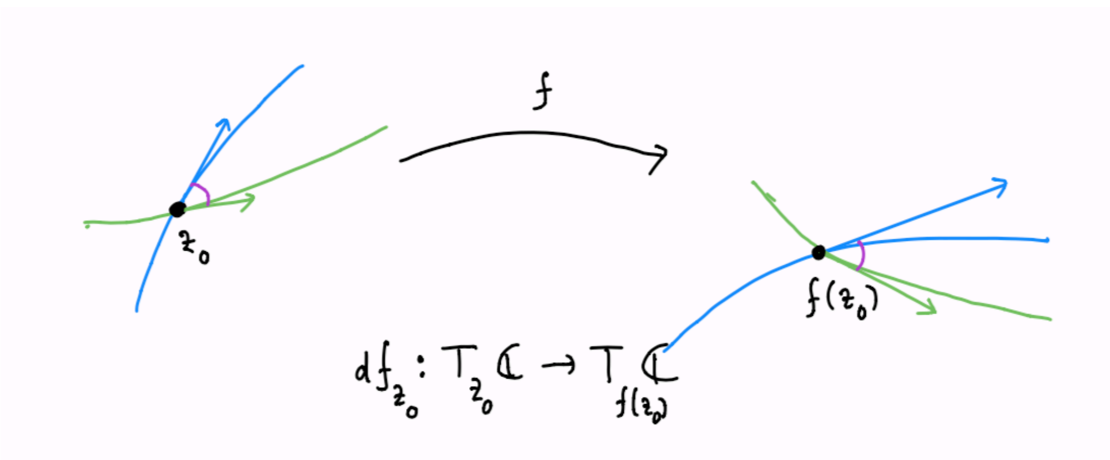
Then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0).$$

In other words, the differential map $df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$$
$$\gamma'(t_0) \to f'(z_0)\gamma'(t_0)$$

converts tangent vectors based at z_0 into ones based at $f(z_0)$, via a rotation amount $arg\, f'(z_0)$ and a scaling by $|f'(z_0)|$.

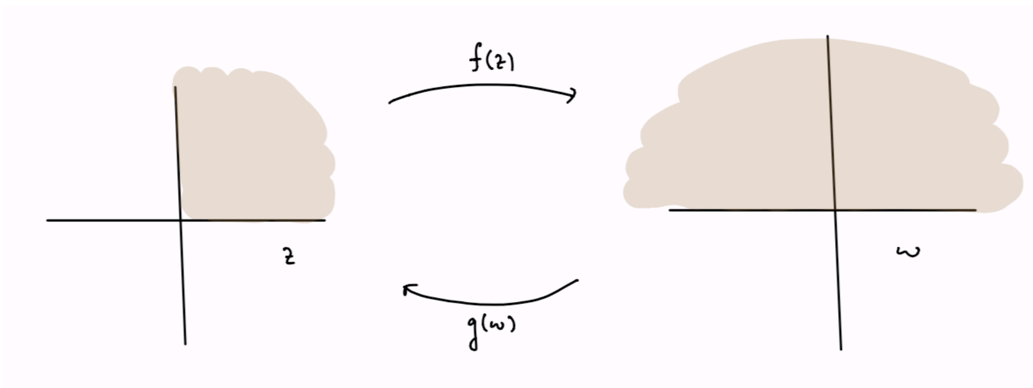


We called a differentiable map $f : A \to \mathbb{C}$ *conformal* on A if this rotation and scaling property holds $\forall\ z \in A$. And we know by the Cauchy-Riemann equations that this rotation-dilation property on tangent vectors is equivalent to $f : A \to \mathbb{C}$ analytic, with $f'(z) \neq 0\ \forall\ z \in A$.

In Chapter 5 we are interested in finding bijective conformal maps between open connected domains $A, B \subseteq \mathbb{C}$. In such cases we will call A and B *conformally equivalent.* Applications include partial differential equations and geometry.

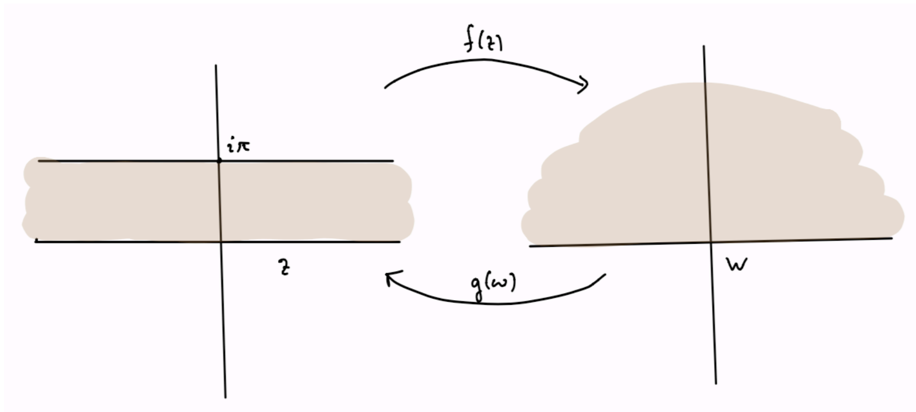
Example: How many conformal bijections can we find between the 1^{st} quadrant and the upper half plane? Just to be clear, we're focusing on the open domains and not worrying about boundary behavior at this point.

$f(z) = z^2$, $g(w) = \sqrt{w}$ is one pair. What others can we find?



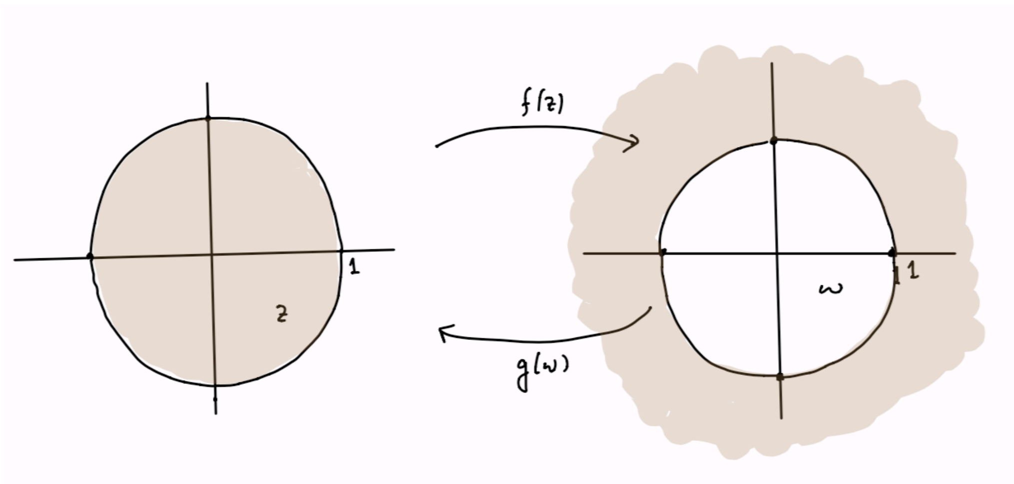
<u>Example</u> Same question for A equal the strip for which $0 < \text{Im}(z) < \pi$, and B equal to the upper half plane.

one pair: $f(z) = e^z$, $g(w) = \log(w)$.



Example Same question for A equal to the open unit disk, B equal to the complement of the closed unit disk.

one pair: $f(z) = \frac{1}{z}, g(w) = \frac{1}{w}$



Chances are that unless our memory is really good we missed some examples .... see the following two versions of the Riemann Mapping Theorem.

<u>Riemann Mapping Theorem</u> (version 1)
Let $A \subseteq \mathbb{C}$ (but $A \neq \mathbb{C}$) be open and simply connected.
Let $z_0 \in A$.
Then $\exists! f: A \rightarrow \mathrm{D}(0; 1)$ such that f is a conformal bijection satisfying

$$f(z_0) = 0$$

$$f'(z_0) \text{ is real and positive.}$$

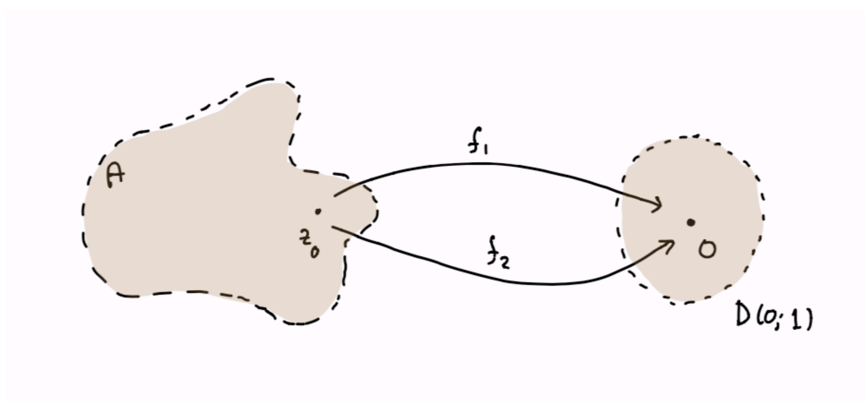
Note that there are three real degrees of freedom for conformal bijections with the disk: 2 from the choice of z_0 and one from the choice of the argument of $f'(z_0)$ (since we can always compose an f with $f(z_0) = 0$ by a rotation about the origin to uniquely make the argument of $f'(z_0)$ whatever we want).

The proof that such maps exist for any open simply connected subset of $\mathbb{C}$ except $\mathbb{C}$ itself would take several lectures to explain and we won't do it in this course. But we already have the tools to prove uniqueness. In fact, these ideas recall class discussions and homework.

proof of uniqueness: Suppose f_1, f_2 satisfy the conditions above. Define

$$g := f_2 \circ f_1^{-1} : \mathrm{D}(0; 1) \rightarrow \mathrm{D}(0; 1).$$

and apply (recall) ideas related to Schwarz's lemma for g and g^{-1} that we applied earlier in the course.

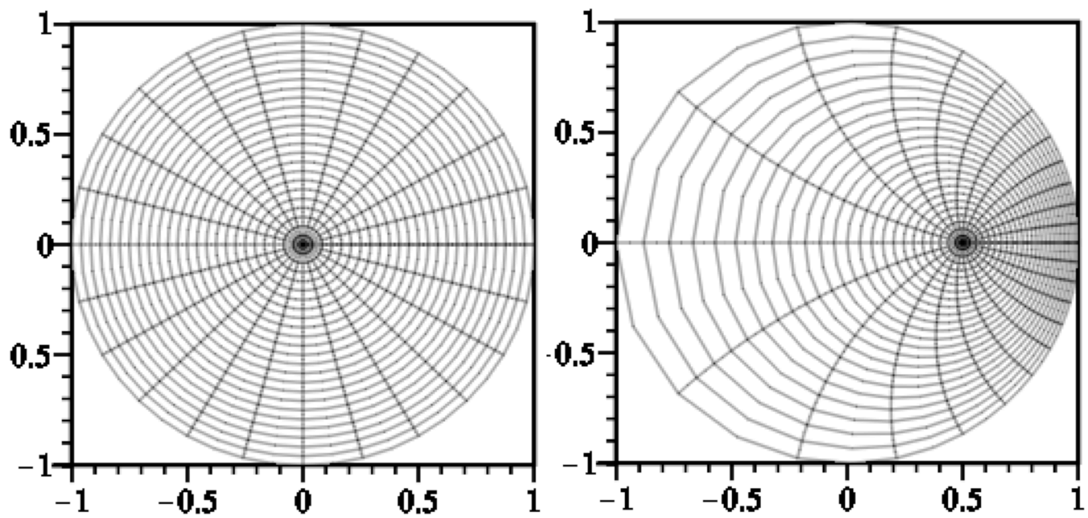


Remark: We actually proved both existence and uniqueness for this version the RMP earlier in the course, using Mobius transformations, for the very special case of

$$f: \mathrm{D}(0; 1) \rightarrow \mathrm{D}(0; 1)$$
$$f(z_0) = 0$$
$$f'(0) \text{ is real and positive.}$$

The mapping is

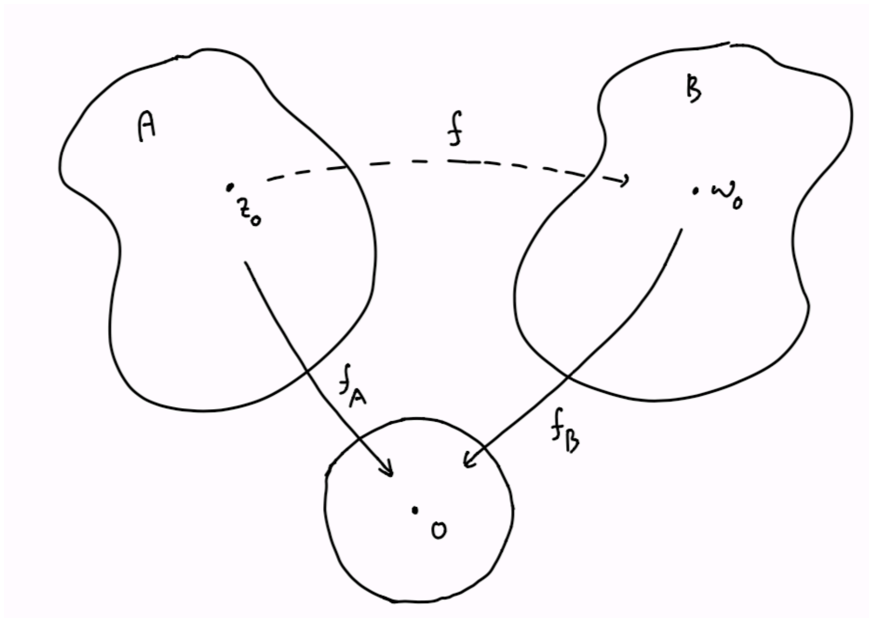
$$f(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$



Riemann Mapping Theorem (version 2)
Let $A, B \subseteq \mathbb{C}$ be open and simply connected but not all of $\mathbb{C}$.
Let $z_0 \in A$, $w_0 \in B$.
Then $\exists! f: A \to B$ such that f is a conformal bijection satisfying

$$f(z_0) = w_0$$
$$f'(z_0) \text{ is real and positive.}$$

proof: Chase the diagram arrows below to prove existence, and then uniqueness, letting f_A, f_B be as in version 1 on the previous page.



The maps we were missing in the examples from the start of class were compositions of the ones we found, with *fractional linear transformations*, of which Mobius transformations are examples.

<u>Def</u> a *fractional linear transformation (FLT)* $f : \mathbb{C} \cap \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function defined by

$$f(z) = \frac{a z + b}{c z + d},$$

where $a, b, c, d \in \mathbb{C}$ and

$$ad - bc = det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 .$$

Note that when the determinant does equal zero, the function f is just a constant. Also, one could normalize the determinant to be $+1$ by dividing all of the coefficients by the same number (a square root of the determinant).

<u>Example</u> $f(z) = a z + b = \dfrac{a z + b}{0 z + 1}$. You will show in your homework that these are the only one-to-one conformal maps defined on all of $\mathbb{C}$. Notice that they are conformal bijections of $\mathbb{C}$.

<u>Exercise</u> Why is there no conformal bijection $f : \mathbb{C} \to \mathrm{D}(0; 1)$? It's a one-line answer if you can think of it.

<u>The algebra of fractional linear transformations</u>: Show that if

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

$$g(w) = \frac{\alpha\,w + \beta}{\gamma\,w + \delta}$$

Then

$$g(f(z)) = \frac{A\,z + B}{C\,z + \mathrm{D}}$$

where

$$\begin{bmatrix} A & B \\ C & \mathrm{D} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

A matrix for the composition FLT can be obtained by multiplying the matrices for the individual FLTs!

Geometers would say: The *group* $SL(2, \mathbb{C})$ ("The *special linear group* of 2×2 matrices with entries from $\mathbb{C}$) *acts* on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The word *special* refers to the fact that each FLT can be represented uniquely with a matrix having determinant exactly equal to 1, and the word *acts* refers to the fact that matrix multiplication (the group operation) in the group, corresponds to composition of transformations on the Riemann sphere.

To be continued .... this algebra makes composing and finding inverses for FLT's straightforward. On Monday we'll continue this discussion and explain interesting geometry related to FLT's.