Math 4200
Wednesday November 20
4.4 Infinite series and infinite partial fractions

Announcements: The section 4.4 introduction is in Monday's notes. We'll start there with infinite series, and continue here with infinite partial fraction expansions.

Theorem 2 If $f(z)$ is analytic on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_k\}$ and if for large $|z|$ there is an M for which

$$\left|f(z)\right| \le \frac{M}{|z|}$$

then

$$\lim_{N\to\infty} \int_{\gamma_N} f(z)\, \pi \cot(\pi z)\, dz = 0.$$

Thus, from Theorem 1,

$$\lim_{N\to\infty} \left(\sum_{\substack{j=-N \\ f \text{ analytic at } j}}^{N} f(j) \right) = -\left(\sum_{\substack{z_k \text{ singular} \\ \text{point off}}} Res\left(f(z)\, \pi \cot(\pi z), z_k\right) \right)$$

proof. Let R be large enough so that all of the singularities of f have modulus less than R. Then f has a Laurent series in the complement of this disk, and because of the decay estimate it only has negative powers of z (because $f\left(\frac{1}{z}\right)$ has a removable singularity at $z = 0$),

$$\begin{aligned} f(z) &= \frac{b_1}{z} + \sum_{m=2}^{\infty} \frac{b_m}{z^m} \\ &= \frac{b_1}{z} + g(z) \end{aligned}$$

where there is a uniform estimate for g for $|z| \ge R$,

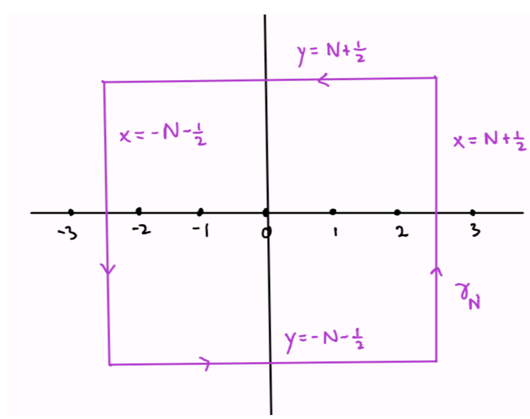
$$|g(z)| \le \frac{C}{|z|^2}.$$

Thus

$$\int_{\gamma_N} f(z)\, \pi \cot(\pi z)\, dz = \int_{\gamma_N} \frac{b_1}{z}\, \pi \cot(\pi z)\, dz + \int_{\gamma_N} g(z)\, \pi \cot(\pi z)\, dz.$$

The integrand in the first integral is even because it's a product of two odd functions but the contour differential dz is odd for this square contour, so the first integral evaluates to zero! And because $|\pi \cot(\pi z)| \le 2$ on the contours γ_N and $g(z)$ decays, the second integral's modulus can be estimated by

$$\left| \int_{\gamma_N} g(z)\, \pi \cot(\pi z)\, dz \right| \le \frac{2C}{N^2} 4 \cdot (2N+1) \to 0 \quad \text{as } N \to \infty.$$



for reference, if for large z we have a decay estimate

$$\left|f(z)\right| \le \frac{M}{|z|}$$

then

$$\lim_{N\to\infty}\left(\sum_{\substack{j=-N \\ f\ analytic\ at\ j}}^{N} f(j)\right) = -\left(\sum_{\substack{z_k\ singular \\ point\ of\ f}} Res\left(f(z)\,\pi\cot(\pi z), z_k\right)\right).$$

Examples 1) $f(z) = \frac{1}{z^{2k}}$, as we discussed in Monday's notes. Magic summation formulas for $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ based soley on coefficients of the Laurent series for $\pi\cot(\pi z)$ at the origin.

2) For $z_0 \in \mathbb{C}\setminus\mathbb{Z}$, $f(z) = \frac{1}{z - z_0}$, which has a simple pole at z_0. So

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n - z_0} = -\,Res\left(\frac{1}{z - z_0}\,\pi\cot(\pi z), z_0\right) = -\pi\cot(\pi z_0).$$

So, replacing z_0 with z, multiplying the equation by -1, and arranging the sum on the left as a sum of two series that converge uniformly on compact subsets that avoid the integers:

$$\pi\cot(\pi z) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z - n}$$

$$\pi\cot(\pi z) = \frac{1}{z} + \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{z-n} + \frac{1}{n} + \sum_{n=1}^{N}\frac{1}{z+n} - \frac{1}{n}\right)$$

Because

$$\frac{1}{z-n} + \frac{1}{n} = \frac{z}{(z-n)n}, \qquad \frac{1}{z+n} - \frac{1}{n} = \frac{z}{(z+n)n}$$

each of the modified subseries converges uniformly on compact subsets that avoid the integers (by the Weirerstrass M test with comparison series the tail of

$$\sum_{n=N}^{\infty}\frac{1}{n^2}\ ),$$

so

$$\pi\cot(\pi z) = \frac{1}{z} + \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{z-n} + \frac{1}{n}\right) + \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{z+n} - \frac{1}{n}\right).$$

This is like an infinite partial fractions decomposition for $\pi\cot(\pi z)$!

3) In your homework you'll prove another infinite partial fractions expansion,

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} .$$

Notice that the pole locations and orders agree on both sides of the equation, so there would be some hope for the identity being true. There is a very quick proof of this identity if you can see it, based on doing something to the last identity on the previous page.

These last two examples illstrate a general theory for infinite sum partial fraction expansions for *meromorphic* functions, i.e. functions which are analytic on $\mathbb{C}$ except for an at most countable set $\{z_k\}$ of isolated pole-type singularities (as opposed to essential singularities).

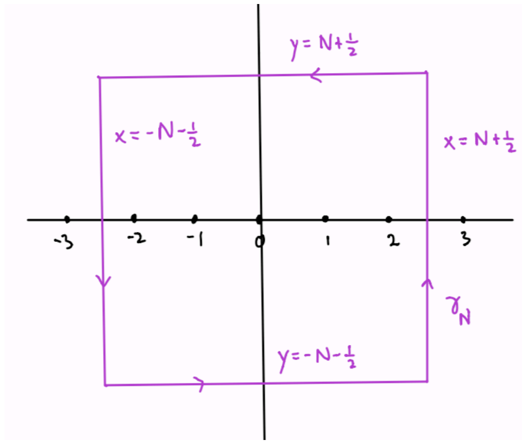
The *Mittag-Leffler Theorem* says you can basically create an infinite sum analytic function with prescribed isolated poles, and with prescribed negative power Laurent series at those poles. (There's a short page at Wikipedia that contains the two identities we've written down, as well as some others.)

Related to infinite sum formulas for Meromorphic functions, there are infinite product functions as well for analytic functions, involving the zeroes of the analytic function. (The connection between the two theories is the logarithm, which takes products to sums.) In Chapter 7 of our text there are infinite product identities related to the Riemann-Zeta function, for example.

Appendix: Uniform bound estimates of $\cot(\pi z)$ on the half-integer contours γ_N. These estimates hold:

$$|\cot(\pi z)| \le 1 \text{ on the vertical paths}$$

$$|\cot(\pi z)| \le 2 \text{ on the vertical paths (the bound limits to 1 as } N \to \infty \text{)}$$



One efficient way to make these estimates is to use the various trig identities we discussed in Chapter 1. For $u, v \in \mathbb{R}$,

$$\cos(u + i v) = \cos(u)\cos(i v) - \sin(u)\sin(i v)$$
$$\cos(u + i v) = \cos(u)\cosh(v) - \sin(u) i \sinh(v)$$

$$\sin(u + i v) = \cos(u)\sin(i v) + \sin(u)\cos(i v)$$
$$\sin(u + i v) = \cos(u) i \sinh(v) + \sin(u)\cosh(v).$$

So,

$$\frac{\cos(u + i v)}{\sin(u + i v)} = \frac{\cos(u)\cosh(v) - i \sin(u)\sinh(v)}{\sin(u)\cosh(v) + i \cos(u) \sinh(v)}$$

$$\left|\frac{\cos(u + i v)}{\sin(u + i v)}\right|^2 = \frac{\cos^2(u)\cosh^2(v) + \sin^2(u)\sinh^2(v)}{\sin^2(u)\cosh^2(v) + \cos^2(u)\sinh^2(v)}.$$

So along the vertical contours, using

$$\cos^2\left(\pm\pi\left(N + \frac{1}{2}\right)\right) = 0, \ \sin^2\left(\pm\pi\left(N + \frac{1}{2}\right)\right) = 1$$

$$\Rightarrow \left|\frac{\cos(\pi x + i \pi y)}{\sin(\pi x + i \pi y)}\right|^2 = \frac{\sinh^2(\pi y)}{\cosh^2(\pi y)} \le 1.$$

And along the horizontal contours, and for $v = \pm\pi\left(N + \frac{1}{2}\right)$, and using

$$\cosh^2(v) - \sinh^2(v) = 1,$$

$$\Rightarrow \left|\frac{\cos(u + i v)}{\sin(u + i v)}\right|^2 = \frac{\cos^2(u)\cosh^2(v) + \sin^2(u)\left(\cosh^2(v) - 1\right)}{\sin^2(u)\left(\sinh^2(v) + 1\right) + \cos^2(u)\sinh^2(v)}$$

$$= \frac{\cosh^2(v) - \sin^2(u)}{\sinh^2(v) + \sin^2(u)} \le \frac{\cosh^2(v)}{\sinh^2(v)} \to 1 \quad \text{as } N \to \infty.$$

claim verified.

Math 4200-001
Week 13 concepts and homework
4.4
Due Wednesday November 27 at start of class.

4.4: 2, 3, 4, 5, 8, 9
5.1: 10, 11, 12.