Math 4200
Friday November 1
3.3 Laurent series.

Announcements: It'll take most (or all) of today to prove the Laurent series theorem - you'll see it's all related to geometric series (like examples Wed.)

Laurent Series Theorem For $0 \le R_1 < R_2$ let

$$A = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f : A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}.$$

$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $\underbrace{|z - z_0| < R_2}_{\text{the big disk}}$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$. ← we did this before

And $S_2(z)$ converges for $\underbrace{|z - z_0| > R_1}_{\text{outside little disk}}$, and uniformly for $|z - z_0| \ge r_1 > R_1$. ← see HW.

Notes: (2) $\Rightarrow$ (1) is immediate from the uniform convergence of $S_1(z) + S_2(z)$ on all compact subannuli $r_1 \le |z - z_0| \le r_2$ with $R_1 < r_1 < r_2 < R_2$. And, the uniform absolute convergence on the restricted domains follows from the convergence on the larger ones.

e.g. $S_1(z)$ converges $\forall\ |z-z_0| < R_2$

Let $r_2 < R_2$.

pick $r_2 < \rho < R_2$.

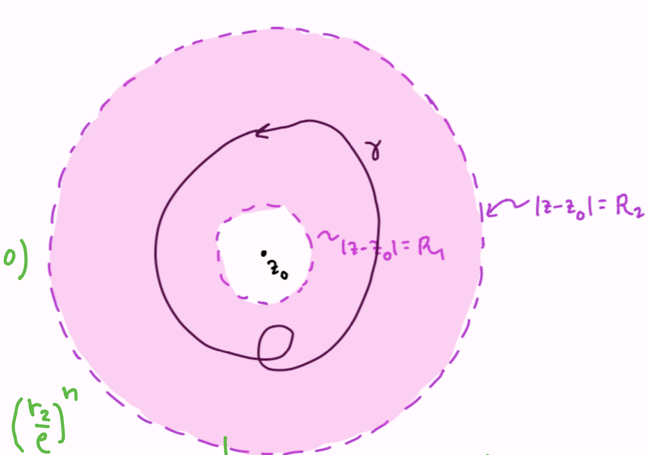
$S_1(z)$ converges for $|z-z_0| = \rho$

$\Rightarrow |a_n|\rho^n \to 0$ (terms $\to 0$)

then if $|z-z_0| \le r_2$, try M-test

$$\sum_{n=0}^{\infty} |a_n||z-z_0|^n \le \sum_{n=0}^{\infty} |a_n| r_2^n = \sum_{n=0}^{\infty} \underbrace{|a_n|\rho^n}_{\le C} \left(\frac{r_2}{\rho}\right)^n \le C \frac{1}{1 - r_2/\rho}$$

because $\frac{r_2}{\rho} < 1$



(3) The Laurent coefficients a_n, b_m are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) (\zeta - z_0)^{m-1} d\zeta.$$

In particular the contour integral of f itself has value

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i\, b_1.$$

For this reason, the coefficient b_1 of $\frac{1}{z - z_0}$ in the Laurent series, is called the *residue* of f at z_0.

proof of (2) $\Rightarrow$ (3) in the Laurent series theorem:

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$$b_m = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta - z_0)^{m-1} d\zeta.$$

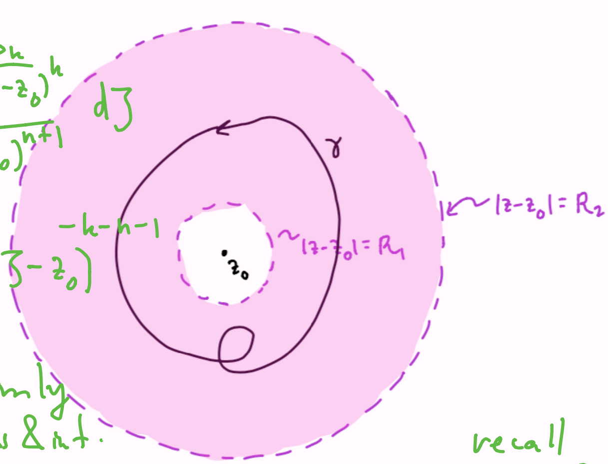
proof: We'll write $f(\zeta) = S_1(\zeta) + S_2(\zeta)$ and just compute the prescribed contour integrals of f, to see how they pick off the individual Laurent coefficients . We'll use the uniform convergence of the series $S_1(\zeta), S_2(\zeta)$ on γ to interchange the integrals with the summations. This is exactly the same philosophy and geometric series ideas as in the examples on Wednesday, but applied in this general context.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \; + \sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k}$$

$n \ge 0$ $\quad a_n \overset{?}{=} \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \;\; ? \qquad b_m \overset{?}{=} \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta - z_0)^{m-1} d\zeta \;\; ?$

$$\frac{1}{2\pi i}\int_\gamma \frac{\sum_{k=0}^{\infty} a_k(\zeta-z_0)^k}{(\zeta-z_0)^{n+1}} d\zeta + \frac{1}{2\pi i}\int_\gamma \frac{\sum_{k=1}^{\infty} \frac{b_k}{(\zeta-z_0)^k}}{(\zeta-z_0)^{n+1}} d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty} a_k(\zeta-z_0)^{k-n-1} d\zeta + \frac{1}{2\pi i}\int_\gamma \sum_{k=1}^{\infty} b_k(\zeta-z_0)^{-k-n-1}$$



series converge uniformly on γ, so exchange sums & int.

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty} a_k \int_\gamma (\zeta-z_0)^{k-n-1} d\zeta + \frac{1}{2\pi i}\sum_{k=1}^{\infty} \int_\gamma (\zeta-z_0)^{-k-n-1} d\zeta \to 0$$

unless $k-n-1=-1$ antiderivs. get 0 ($k=n$!)

$p \le -2$ antiderivs

recall $(\zeta-z_0)^p$ has antideriv $\frac{1}{p+1}(\zeta-z_0)^{p+1}$ $\forall p \in \mathbb{Z}$ except $p=-1$

only nonzero term is $\frac{1}{2\pi i} a_n \underbrace{\int_\gamma \frac{1}{\zeta - z_0} d\zeta}_{2\pi i} = a_n$ ✓ $\left(I(\gamma; z_0) = 1 \right)$.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k}$$

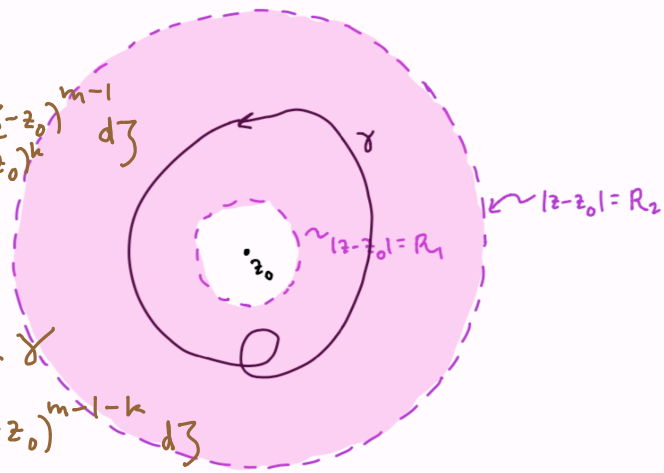
$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad ? \qquad b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta) (\zeta - z_0)^{m-1} d\zeta \quad ?$$

$m \geq 1$

$$b_m \overset{?}{=} \frac{1}{2\pi i} \int_\gamma \sum_{k=0}^{\infty} a_k (\zeta - z_0)^k (\zeta - z_0)^{m-1}$$

analytic in $D(z_0; R_1)$ so int $= 0$ by Cauchy's Thm.

$$+ \frac{1}{2\pi i} \int_\gamma \sum_{k=1}^{\infty} b_k \frac{(\zeta - z_0)^{m-1}}{(\zeta - z_0)^k} d\zeta$$



$\|$ unif conv on γ

$$b_m \overset{?}{=} \frac{1}{2\pi i} \sum_{k=1}^{\infty} b_k \int_\gamma (\zeta - z_0)^{m-1-k} d\zeta$$

$= 0$ except for $k = m$ then we get $2\pi i$

$$b_m \overset{\checkmark}{=} \frac{1}{2\pi i} b_m 2\pi i$$ ✓

proof of (1) $\Rightarrow$ (2) in the Laurent series theorem:

<u>Laurent Series Theorem</u> For $0 \leq R_1 < R_2$ let

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be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

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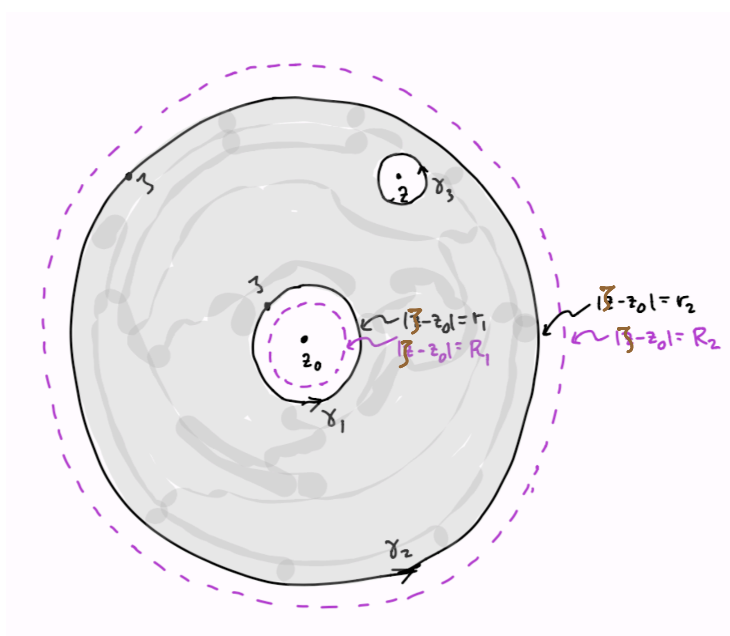
Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for and compact subdisk $|z - z_0| \leq r_2 < R_2$.
And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly absolutely for any complement of a strictly larger disk, $|z - z_0| \geq r_1 > R_1$.

proof: We'll just focus on the convergence statements, because the absolute convergence statements follow from those. Let z be in the open annulus A. Pick r_1, r_2, ε so that $R_1 < r_1 < r_2 < R_2$ and so that all points of $\bar{D}(z_0; \varepsilon)$ lie in the sub-annulus $r_1 < |z - z_0| < r_2$. See figure. Let γ_1 be the circle of radius r_1 about z_0; let γ_2 be the circle of radius r_2 about z_0; let γ_3 be the circle of radius ε about z. All circles oriented counterclockwise as usual.



Then by the Green's Theorem version of Cauchy's Theorem (for domains with holes),

$$\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad .$$

By C.I.F. on the little disk bounded by γ_3,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\,d\zeta\,,$$

and substituting this into the formula above yields

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

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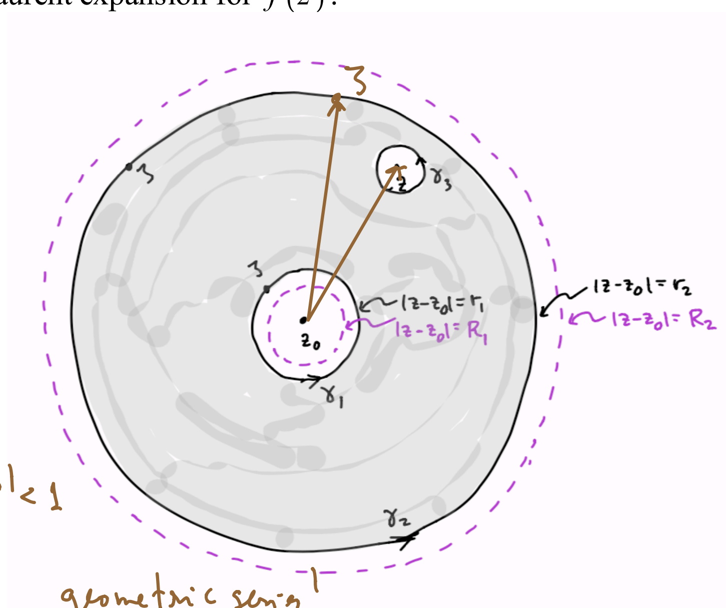
Use our geometric series wizardry to find the Laurent expansion for $f(z)$.



$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\underbrace{(\zeta - z_0)}_{\text{big modulus}} - (z - z_0)}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0}\,\frac{1}{\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}\,d\zeta$$

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{r_2} < 1$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

geometric series!

$$= \frac{1}{2\pi i}\int_{\gamma_2} \sum_{n=0}^{\infty} \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}\,d\zeta$$

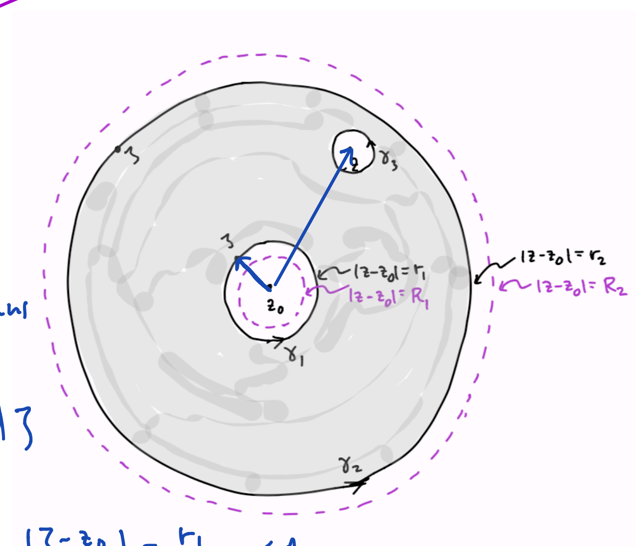
unif conv.

$$= \cancel{\frac{1}{2\pi i}} \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)\cancel{(z - z_0)^n}}{(\zeta - z_0)^{n+1}}\,d\zeta$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta - z_0) - \underbrace{(z - z_0)}_{\text{big } \cancel{\text{mo}} \text{ modulus term}}}\,d\zeta$$



$$= -\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{-(z-z_0)}\,\frac{1}{1-\left(\frac{\zeta - z_0}{z - z_0}\right)}\,d\zeta$$

$$\left|\frac{\zeta - z_0}{z - z_0}\right| = \frac{r_1}{|z - z_0|} < 1$$

$$= \frac{1}{2\pi i}\,\frac{1}{z - z_0}\int_{\gamma_1}\underbrace{\sum_{k=0}^{\infty} f(\zeta)\left(\frac{\zeta - z_0}{z - z_0}\right)^k}_{\text{unif conv.}}\,d\zeta$$

$$= \frac{1}{2\pi i}\,\frac{1}{z - z_0}\sum_{k=0}^{\infty}\frac{1}{(z-z_0)^k}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^k\,d\zeta$$

$$= \sum_{k=0}^{\infty}\cancel{\tfrac{1}{2\pi i}}\,\frac{1}{(z - z_0)^{k+1}}\,\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^k\,d\zeta$$

$$= \sum_{m=1}^{\infty}\frac{1}{(z - z_0)^m}\underbrace{\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^{m-1}\,d\zeta}_{b_m\text{'s}}$$

$k+1=m$

Isolated singularities table.

Let f be analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$.

type of singularity at z_0	Laurent series definition	characterization in terms of $\lim_{z\to z_0} f(z)$
<u>removable</u> (because f extends to be analytic at z_0)	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in L.S.)	any of: ① $\lim_{z\to z_0} f(z)=L\in\mathbb{C}$ exists ② $\lvert f(z)\rvert\le M$ $\forall\ 0<\lvert z-z_0\rvert\le\rho$ for some $0<\rho<r$. ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$.
<u>pole</u> (North pole!) <u>of order N</u> <u>simple pole</u> if $N=1$	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{N}\frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$	① $\lim_{z\to z_0} f(z)=\infty$ (the north pole on the Riemann sphere) or ② $\exists\ N$ s.t. $g(z)=(z-z_0)^N f(z)$ has a removable singularity @ $z=z_0$, with $g(z_0)\neq 0$.
<u>essential singularity</u>	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^m + \sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}$ with $\{m_j\}\to\infty$, $b_{m_j}\neq 0$	(Monday) $\boxed{\forall\ 0<\rho<r \quad \overline{f(D(z_0;\rho)\setminus\{z_0\})}=\mathbb{C}\ !}$ (In fact, more is true and is called "Picard's Theorem: $f(D(z_0;\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$ except for <u>at most a single point</u>!) e.g. $f(z)=e^{1/z}$ @ $z_0=0$ $f(D(0;\rho)\setminus\{0\})=\mathbb{C}\setminus\{1\}$ $\forall\rho>0$