Math 4200
Monday November 18
4.3-4.4 Integral applications of the residue theorem, including infinite series magic.

Announcements:

reminder: HW for Wednesday November 20
4.3: 1, 2, 4, 6, 10, 14, 17, 20ab.
There are a lot of good worked examples in the text. In problem 6 you may use entry #5 on the Definite integral table 4.3.1, page 296. The text explains why this table entry is true on pages 289-293 and summarizes it as Proposition 4.3.16. It uses an interesting contour around a branch domain for the logarithm. For problem 14, use the ideas and contours of Example 4.3.18.   Comments on following pages.

After finishing the example from Friday, and discussing 4.3.4, 4.3.16 a bit more, we'll move on to section 4.4 fun.

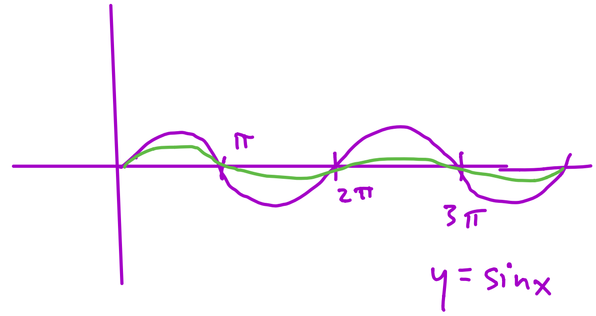
plan. W 4.4
F
M   5.1 - 5.2   conformal mapping.
W
Thanksgiving.
M
W   presentations ; we'll probably also need an hour outside of class.
- - - - classes end
F ← reading day

M
W
F ← final exam, also when project reports are due.

finish from Friday

<u>Example</u> (Relates to homework problem 4.3.2). Show

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2},$$

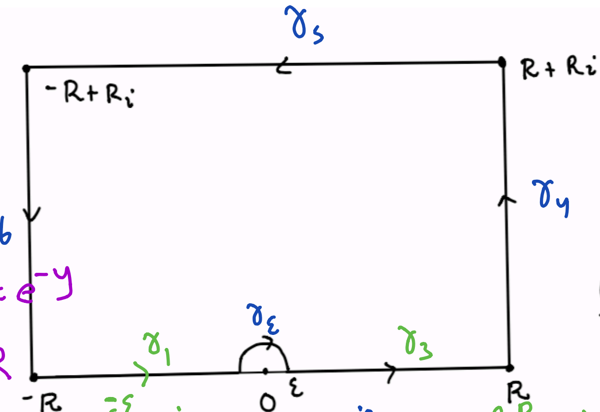


using

$$\int_{\gamma_{\varepsilon,R}} \frac{e^{iz}}{z}\,dz$$

Note, this improper integral does not converge absolutely, but converges conditionally by the alternating series test....and also, we use an interesting contour and "principal value" techniques to evaluate it.

Monday



Let $R\to\infty$.
Then let $\varepsilon \to 0$

Monday these $\to 0$ as $R \to \infty$

$$\left|\int_{\gamma_4} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^R \frac{e^{i(R+iy)}}{R+iy}\,i\,dy\right|$$

$z = R + iy$
$0 \le y \le R$
$dz = i\,dy$

$$\left|e^{i(R+iy)}\right| = e^{-y}$$

$$|R+iy| \geq R$$

$$\le \int_0^R |\ ||\ |$$

$$\le \int_0^R \frac{e^{-y}}{R}\,dy$$

$$= \frac{1}{R}\left(-e^{-y}\right)\Big]_0^R$$

$$= \frac{1}{R}\left(-e^{-R}+1\right) \le \frac{1}{R} \to 0 \text{ as } R\to\infty.$$

also $\left|\int_{\gamma_6} \frac{e^{iz}}{z}\,dz\right| \to 0$ as $R \to \infty$.

$$\left|\int_{\gamma_5} \frac{e^{iz}}{z}\,dz\right| \le \int_{\gamma_5} \frac{|e^{iz}|}{R}\,|dz| \le \frac{e^{-R}}{\cancel{R}}\,2\cancel{R} \to 0$$

as $R \to \infty$.

on γ_5 $\left|e^{iz}\right| = \left|e^{i(x+iR)}\right| = e^{-R}$.

$$0 = \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz + \int_\varepsilon^R \frac{e^{ix}}{x}\,dx + \int_{\gamma_4+\gamma_5+\gamma_6} \frac{e^{iz}}{z}\,dz$$

no singularities inside contour

$$\int_{-R}^{-\varepsilon} \frac{\cos x}{x} + \frac{i\sin x}{x}\,dx + \int_\varepsilon^R \frac{\cos x}{x} + \frac{i \sin x}{x}\,dx$$

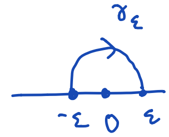
odd integrand

cancel, even though individually blow up as $\varepsilon \to 0$.

$$= 2i\int_\varepsilon^R \frac{\sin x}{x}\,dx$$

even integrand.

Let $R \to \infty$: $2i\int_{\varepsilon}^{\infty} \frac{\sin x}{x}\,dx + \underbrace{\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz}_{} = 0$



$$\frac{e^{iz}}{z} = \frac{1 + iz - \frac{z^2}{2!} + \cdots}{z}$$

$= \frac{1}{z} + g(z)$ $\quad g$ analytic (hence bd) near 0.

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz = \int_{\gamma_\varepsilon} \frac{1}{z}\,dz + \underbrace{\int_{\gamma_\varepsilon} g(z)\,dz}_{}$$

as $\varepsilon \to 0$, $\left|\quad\right| \leq \pi \varepsilon M \to 0.$

bound on $|g(z)|$ near 0

$$= -\int_0^{\pi} \frac{1}{\varepsilon \cancel{e^{i\theta}}}\, \varepsilon i \cancel{e^{i\theta}}\,d\theta$$

$z = \varepsilon e^{i\theta}$

$dz = \varepsilon i e^{i\theta}\,d\theta$

$= -\pi i$

$\varepsilon \to 0$

$$2i\int_0^{\infty} \frac{\sin x}{x}\,dx - \pi i = 0.$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

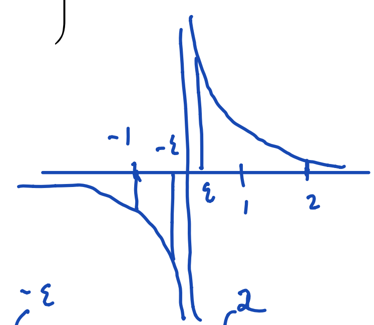
In the previous exercise $\dfrac{e^{iz}}{z}$ has a singularity at $z = 0$ even though $\dfrac{\sin(x)}{x}$ is continuous at $x = 0$.
There is a general class of integrals, called *Principal Value* (or *PV*) integrals, that one can compute, even when the actual integral doesn't exist. These PV integrals are often important in e.g. physics, I think.

<u>Def</u> If f is continous on $[a, b]$ except at $x_0 \in (a, b)$ then

$$PV\left(\int_a^b f(x)\,\mathrm{d}x\right) := \lim_{\varepsilon \to 0}\left(\int_a^{x_0-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_0+\varepsilon}^{b} f(x)\,\mathrm{d}x\right)$$

provided the limit exists.

$y = \frac{1}{x}$



<u>Example</u>

$$PV\left(\int_{-1}^{2} \frac{1}{x}\,\mathrm{d}x\right) = \ln(2)$$

even though

$$\int_{-1}^{0} \frac{1}{x}\,\mathrm{d}x = -\infty, \quad \int_{0}^{2} \frac{1}{x}\,\mathrm{d}x = +\infty .$$

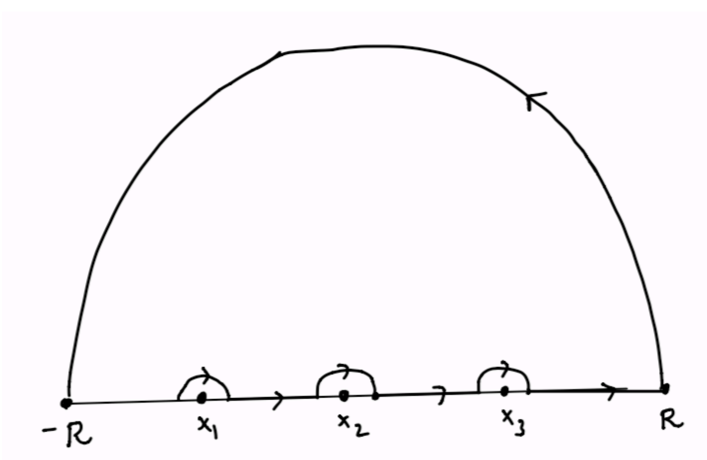
$$\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{2} \frac{1}{x}\,dx$$

$$= \ln|x|\Big]_{-1}^{-\varepsilon} + \ln x\Big]_{\varepsilon}^{2}$$

$$= \cancel{\ln \varepsilon} - \cancel{\ln 1} + \ln 2 - \cancel{\ln \varepsilon}$$

$$= \ln 2$$

Using principal value ideas one can often compute $PV\left(\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right)$ using contours like the one below.
This is Proposition 4.3.11 in the text, of which our worked example was an instance.



Mellin transform

4.3.6 $\int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx \qquad 0<\alpha<3$

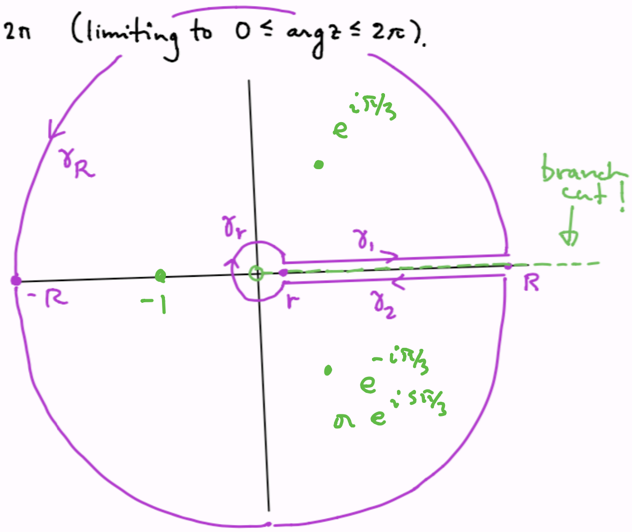
so that improper integral converges at 0

so that improper integral converges at ∞.

$$\boxed{f(z)=\frac{z^{\alpha-1}}{1+z^3}=\frac{e^{(\alpha-1)\log z}}{1+z^3}.}$$

Good branch for logarithm: $0<\arg z<2\pi$ (limiting to $0\le \arg z\le 2\pi$).

& contours $\gamma=\gamma_1+\gamma_R+\gamma_2+\gamma_3$



Simple Poles of f inside γ?
$z^3=-1$: $e^{i\pi/3}, -1, e^{-i\pi/3}$

Res : $\frac{g(z_0)}{h'(z_0)}$

Key point: $\int_{\gamma_1} f(z)\,dz+\int_{\gamma_2} f(z)\,dz$

combine rather than cancel because

$$f(z)=\frac{|z|^{\alpha-1}e^{i(\alpha-1)\arg z}}{1+z^3}$$

and $\arg z=0$ on γ_1
$\arg z=2\pi$ on γ_2.

- can show $\left|\int_{\gamma_R} f(z)\,dz\right|\to 0$ as $R\to\infty$.

$\left|\int_{\gamma_r} f(z)\,dz\right|\to 0 \quad r\to 0$.

$$\int_{\gamma_1} f(z)\,dz=\int_r^R \frac{x^{\alpha-1}}{1+x^3}\,dx.$$

$$\int_{\gamma_2} f(z)\,dz=-\int_r^R \frac{e^{(\alpha-1)\log z}}{1+x^3}\,dx \quad (z \to x)$$

$$\frac{e^{(\alpha-1)(\ln|x|+i2\pi)}}{1+x^3}\,dx$$

$$=\frac{x^{\alpha-1}e^{i(\alpha-1)2\pi}}{1+x^3}\,dx$$

$$\int_{\gamma_2} f(z)\,dz=e^{i(\alpha-1)2\pi}(-1)\int_r^R\frac{x^{\alpha-1}}{1+x^3}\,dx$$

put it all together

$\int_{\gamma_1+\gamma_2} f(z)\,dz$ = a multiple of what we want (after $r\to0, R\to\infty$)

$\|$ integrals over $\gamma_r, \gamma_R\to 0$ as $r\to 0$, $R\to\infty$.

$2\pi i\left(\sum \text{Res}(f;z_j)\right)$.

Wow!

Using a contour like on the previous page wouldn't work for this problem - the real parts of the two key contour integrals *would* cancel out for computing this integral on $[0, \infty]$. But the following contour does work:

4.3.14 $\displaystyle\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx$

$f(z) = \dfrac{\log z}{(z^2+1)^2}$

verify and use:

$$\mathrm{Re}\int_{\gamma_2} f(z)\,dz = \mathrm{Re}\int_{\gamma_1} f(z)\,dz$$

(and each limit to the integral we wish to find.)