Math 4200
Monday November 18
4.3-4.4 Integral applications of the residue theorem, including infinite series magic.

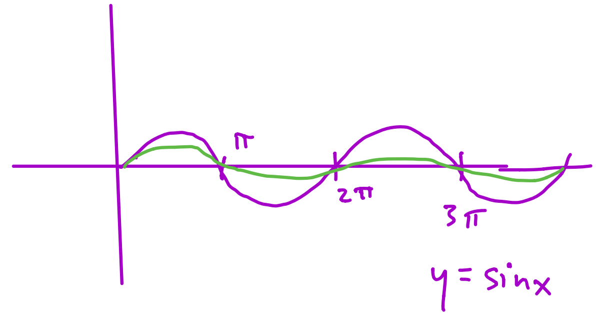
Announcements:

reminder: HW for Wednesday November 20
4.3: 1, 2, 4, 6, 10, 14, 17, 20ab.
There are a lot of good worked examples in the text. In problem 6 you may use entry #5 on the Definite integral table 4.3.1, page 296. The text explains why this table entry is true on pages 289-293 and summarizes it as Proposition 4.3.16. It uses an interesting contour around a branch domain for the logarithm. For problem 14, use the ideas and contours of Example 4.3.18. Comments on following pages.

After finishing the example from Friday, and discussing 4.3.4, 4.3.16 a bit more, we'll move on to section 4.4 fun.

Example (Relates to homework problem 4.3.2). Show

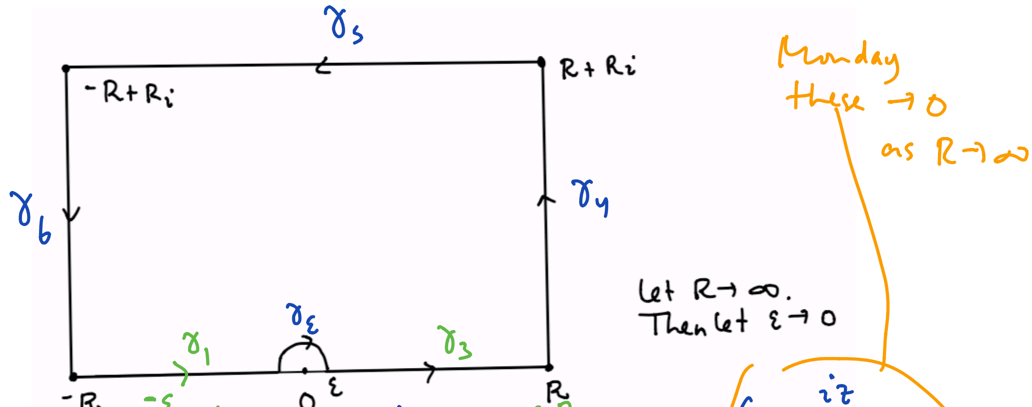
$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2},$$



using

$$\int_{\gamma_{\varepsilon,R}} \frac{e^{iz}}{z}\, dz$$

Note, this improper integral does not converge absolutely, but converges conditionally by the alternating series test....and also, we use an interesting contour and "principal value" techniques to evaluate it.



$$0 = \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\, dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{\gamma_4+\gamma_5+\gamma_6} \frac{e^{iz}}{z}\, dz$$

no singularities inside contour

$$\int_{-R}^{-\varepsilon} \frac{\cos x}{x} + \frac{i \sin x}{x}\, dx + \int_{\varepsilon}^{R} \frac{\cos x}{x} + \frac{i \sin x}{x}\, dx$$

odd integrand: cancel, even though individually blow up as ε → 0.

$$= 2i \int_{\varepsilon}^{R} \frac{\sin x}{x}\, dx$$

even integrand.

In the previous exercise $\frac{e^{iz}}{z}$ has a singularity at $z = 0$ even though $\frac{\sin(x)}{x}$ is continuous at $x = 0$.
There is a general class of integrals, called *Principal Value* (or *PV*) integrals, that one can compute, even when the actual integral doesn't exist. These PV integrals are often important in e.g. physics, I think.

<u>Def</u> If f is continous on $[a, b]$ except at $x_0 \in (a, b)$ then

$$PV\left(\int_a^b f(x)\,dx\right) := \lim_{\varepsilon \to 0}\left(\int_a^{x_0 - \varepsilon} f(x)\,dx + \int_{x_0 + \varepsilon}^b f(x)\,dx\right)$$

provided the limit exists.

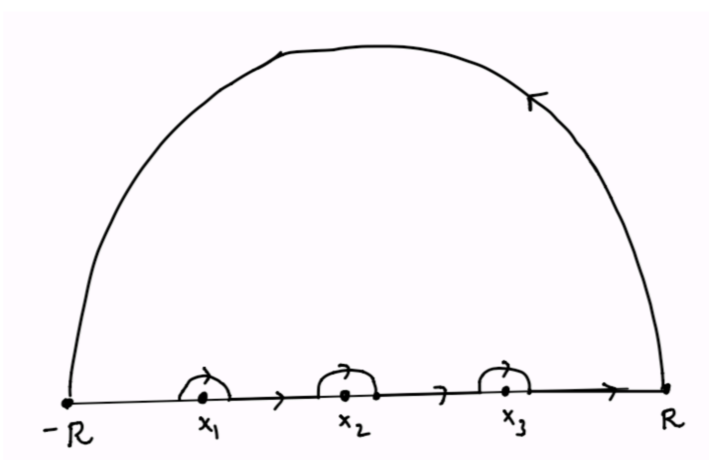
<u>Example</u>

$$PV\left(\int_{-1}^{2} \frac{1}{x}\,dx\right) = \ln(2)$$

even though

$$\int_{-1}^{0} \frac{1}{x}\,dx = -\infty, \quad \int_{0}^{2} \frac{1}{x}\,dx = +\infty\ .$$

Using principal value ideas one can often compute $PV\left(\int_{-\infty}^{\infty} f(x)\,dx\right)$ using contours like the one below.
This is Proposition 4.3.11 in the text, of which our worked example was an instance.



4.3.6 $\int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx$ $\qquad 0 < \alpha < 3$

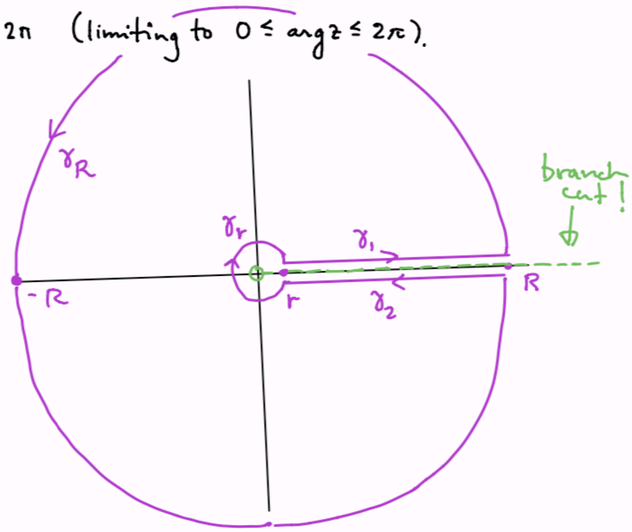
↑ so that improper integral converges at 0 (for $0 < \alpha$)

so that improper integral converges at ∞ (for $\alpha < 3$).

$$f(z) = \frac{z^{\alpha-1}}{1+z^3} = \frac{e^{(\alpha-1)\log z}}{1+z^3}.$$

Good branch for logarithm: $0 < \arg z < 2\pi$ (limiting to $0 \le \arg z \le 2\pi$).

& contours $\gamma = \gamma_1 + \gamma_R + \gamma_2 + \gamma_3$



Key point: $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$

combine rather than cancel because

$$f(z) = \frac{|z|^{\alpha-1} e^{i(\alpha-1)\arg z}}{1+z^3}$$

and $\arg z = 0$ on γ_1

$\arg z = 2\pi$ on γ_2.

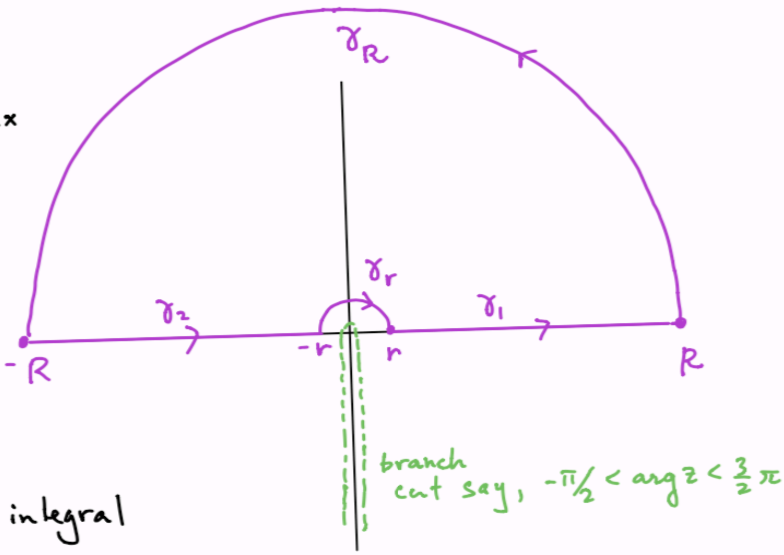
Using a contour like on the previous page wouldn't work for this problem - the real parts of the two key contour integrals *would* cancel out for computing this integral on $[0, \infty]$. But the following contour does work:

4.3.14 $\displaystyle\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx$

verify and use:

$$\mathrm{Re}\int_{\gamma_2} f(z)\,dz = \mathrm{Re}\int_{\gamma_1} f(z)\,dz$$

(and each limit to the integral we wish to find.)



4.4: Infinite series magic and infinite partial fractions, via contour integration....we'll introduce this section today and finish it on Wednesday.

Suppose we have an analytic $f(z)$, f analytic on $\mathbb{C} \setminus \{z_1, z_2, ..., z_k\}$. Suppose we wish to compute

$$\sum_{n=1}^{\infty} f(n).$$

Here's an approach that often works. For example, we'll see that it works in the cases of $f(z) = \frac{1}{z^k}$ where k is an even number, and gives closed form expressions for

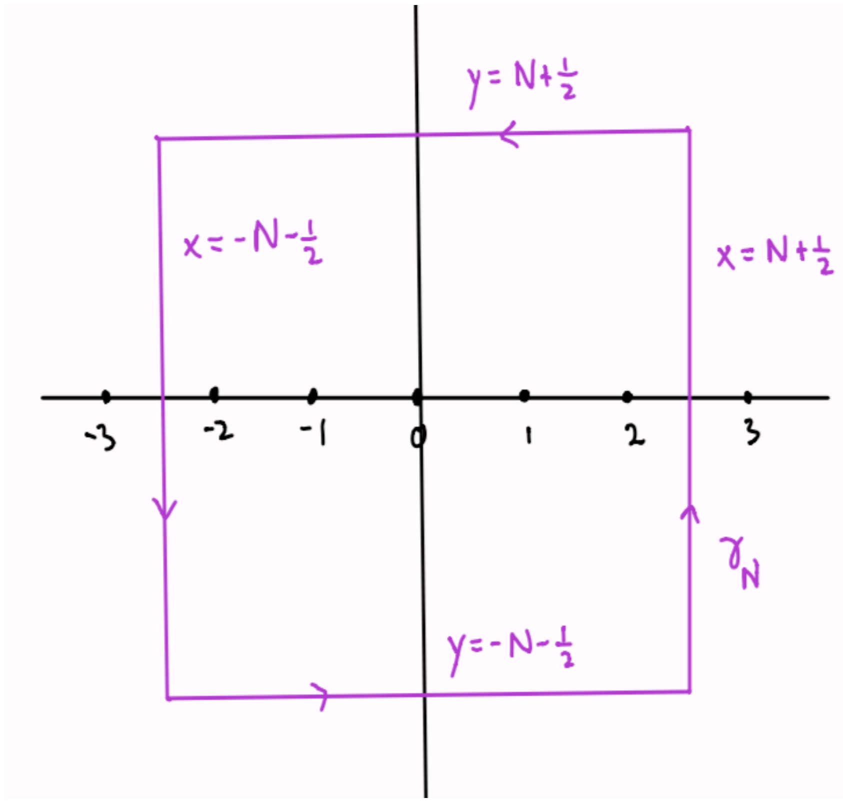
$$\sum_{n=1}^{\infty} \frac{1}{n^2},\ \sum_{n=1}^{\infty} \frac{1}{n^4},\ \sum_{n=1}^{\infty} \frac{1}{n^6},\ \ ....$$

Consider the auxillary function $g(z) = f(z)\pi \cot(\pi z)$. We choose to multiply by $\pi \cot(\pi z)$ partly because

<u>Check</u>: $\pi \cot(\pi z)$ has simple poles precisely at each $n \in \mathbb{Z}$, with residue 1, so $f(z)\pi \cot(\pi z)$ has residue $f(n)$ when f is analytic at $z = n$.

$$f(z)\pi \cot(\pi z) = \frac{f(z)\, \pi \cos(\pi z)}{\sin(\pi z)}.$$

Now consider these special square contours γ_N, as $N \to \infty$. They are chosen so that $|\cot(\pi z)|$ is uniformly bounded (by $M = 2$, for example), as $N \to \infty$.



Theorem 1 Suppose $|f(z)|$ decays sufficiently rapidly so that

$$\lim_{N \to \infty} \int_{\gamma_N} f(z)\, \pi \cot(\pi z)\, dz = 0 .$$

Then

$$\lim_{N \to \infty} \left(\sum_{\substack{j=-N \\ f \text{ analytic at } j}}^{N} f(j) \right) = -\left(\sum_{\substack{z_k \text{ singular} \\ \text{point of } f}} Res\left(f, z_k\right) \right).$$

proof:

Theorem 1 Suppose $|f(z)|$ decays sufficiently rapidly so that

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Then

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Example Find formulas for

$$\sum_{n=1}^{\infty}\frac{1}{n^2},\ \sum_{n=1}^{\infty}\frac{1}{n^4}$$

using $f(z) = \frac{1}{z^2}, \frac{1}{z_4}$. What goes wrong if you try to find a closed form expression for

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\ ?$$

Hint: Here's the beginning of the Laurent series for $\pi\cot(\pi z)$ at the origin:

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> series(π·cot(π·z), z = 0, 12);
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$$z^{-1} - \frac{1}{3}\pi^2 z - \frac{1}{45}\pi^4 z^3 - \frac{2}{945}\pi^6 z^5 - \frac{1}{4725}\pi^8 z^7 - \frac{2}{93555}\pi^{10} z^9 - \frac{1382}{638512875}\pi^{12} z^{11} + O\left(z^{13}\right) \qquad \textbf{(1)}$$

To be continued...