Math 4200
Friday November 15
4.3 Introduction to integral applications of the residue theorem.

With modern technology that can compute definite and indefinite integrals way beyond Calculus, this topic may seem a little out of date. But it turns out there are integrals that are interesting and computable via the Residue Theorem. We'll see some representative examples today, selecting from the large collection in section 4.3. On Monday we'll see magic formulas for certain types of infinite series (section 4.4). These techniques also lead to interesting infinite sum and infinite product formula for various meromorphic functions on $\mathbb{C}$. Infinite product formulas are related to infinite sum formulas via the logarithm, and some of these infinite products are related to the gamma function and the Riemann Zeta function (chapter 7). In a completely different direction of ideas, there is a contour integral formula for the inverse Laplace transform, 8.1-8.2, and contour integrals are often used to understand aspects of the Fourier transform.

Announcements:

reminder: HW for Wednesday November 20
4.3: 1, 2, 4, 6, 10, 14, 17, 20ab.
There are a lot of good worked examples in the text. We'll work some similar problems today.

On Monday we started talking about section 4.3, and discussed how integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\ d\theta$$

can be evaluated via contour integration, where $z = \gamma(\theta) = e^{i\theta}$, $0 \le \theta \le \pi$ is the unit circle:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\ d\theta = \int_{|z|=1} f\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{iz}.$$

For example, we showed that

$$\int_0^{\pi} \cos^4\theta\ d\theta = \frac{3}{8}\pi.$$

Your homework problems 10, 20a are in this vein.

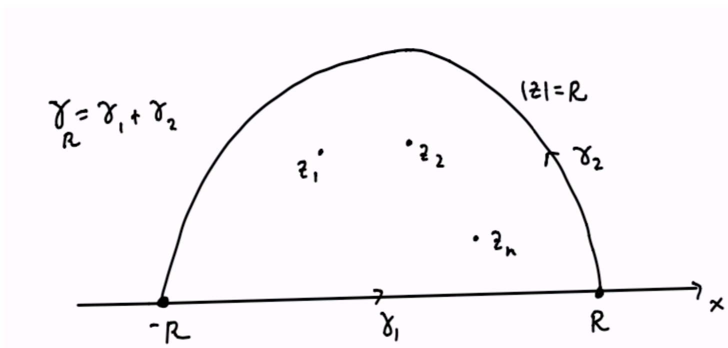
Here's another class of integrals we can compute via contour integration:

<u>Theorem</u> If $f(x)$ is the restriction to the real line of function $f(z)$ which is analytic on all of $\mathbb{C}$ except for a finite number of isolated singularities, none of which occur on the real line; and if for large $|z|$ there is a uniform modulus bound

$$|f(z)| \le \frac{M}{|z|^2}$$

in the closed upper half plane $\{z \mid \mathrm{Re}(z) \ge 0\}$, then

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum \{\text{residues of } f \text{ in the upper half plane}\}$$



proof: Consider $\gamma_R = \gamma_1 + \gamma_2$, apply the Residue Theorem, and let $R \to \infty$. Make good estimates.
(Note, there is an analgous theorem and formula using a semi-circular contour in the lower half plane.)

Example (I stole this from the wikipedia page on the Residue Theorem. Also relates to your homework problem 4.3.17) These integrals arise in probability theory: Show that for $b \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(b\,x)}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{e^{i\,b\,x}}{x^2+1}\,dx = \pi\,e^{-b}$$

First check that the method of the previous page fails for the function you might try first, $f(z) = \dfrac{\cos(b\,z)}{z^2+1}$, when $b > 0$....in both the upper and lower half plane.

<u>Example</u> (Relates to homework problem 4.3.4) Compute

$$\int_0^\infty \frac{1}{x^4 + 1}\, dx = \frac{\pi}{2\sqrt{2}}$$

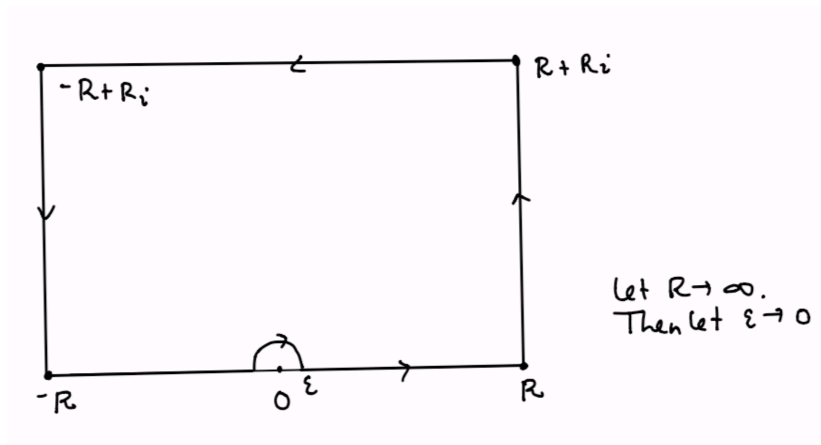
Example (Relates to homework problem 4.3.2). Show

$$\int_0^{\infty} \frac{\sin(x)}{x}\, dx = \frac{\pi}{2},$$

using

$$\int_{\gamma_{\varepsilon,\, R}} \frac{e^{i z}}{z}\, dz$$

Note, this improper integral does not converge absolutely, but converges conditionally by the alternating series test....and also, we use an interesting contour and "principal value" techniques to evaluate it.



In the previous exercise $\frac{e^{iz}}{z}$ has a singularity at $z = 0$ even though $\frac{\sin(x)}{x}$ is continuous at $x = 0$.
There is a general class of integrals, called *Principal Value* (or *PV*) integrals, that one can compute, even when the actual integral doesn't exist. These PV integrals are often important in e.g. physics, I think.

<u>Def</u> If f is continous on $[a, b]$ except at $x_0 \in (a, b)$ then

$$PV\left(\int_a^b f(x)\,dx\right) := \lim_{\varepsilon \to 0}\left(\int_a^{x_0 - \varepsilon} f(x)\,dx + \int_{x_0 + \varepsilon}^b f(x)\,dx\right)$$

provided the limit exists.

<u>Example</u>

$$PV\left(\int_{-1}^2 \frac{1}{x}\,dx\right) = \ln(2)$$

even though

$$\int_{-1}^0 \frac{1}{x}\,dx = -\infty, \quad \int_0^2 \frac{1}{x}\,dx = +\infty .$$

Using principal value ideas one can often compute $PV\left(\int_{-\infty}^{\infty} f(x)\,dx\right)$ using contours like the one below.

This is Proposition 4.3.11 in the text.