Math 4200-001
Week 11 concepts and homework
4.1-4.2
Due Wednesday November 13 at start of class.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 8, 9, 13, 15.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)

Math 4200
Monday November 11

4.3 Integral applications of the residue theorem; topics for exam 2.

HW for Wednesday November 20
4.3: 1, 2, 4, 6, 10, 14, 17, 20ab.

Announcements:

LCB 219

Review session 4:00-5:30 this afternoon - room TBA in class. ~~If I haven't heard back from scheduling~~ by ~~then we'll try JTB 120, which was our review room last time and doesn't have anything scheduled in it yet as of this morning. (And JTB 110 can be our fall-back room.)~~ We'll go over the 2011 exam.
I'll bring copies.

If you're pressed for time you may hand in the homework for Wednesday on Friday instead. The Wednesday exam will include this material.

- homework questions?
- please pick up graded HW.

some partial fractions magic.

$$f(z) = \frac{g(z)}{z-z_0} \qquad g \text{ analytic} \qquad Res(f(z); z_0) = g(z_0)$$

$$= \frac{g(z_0) + g'(z_0)(z-z_0) + \cdots}{(z-z_0)}$$

partial frac: $\boxed{= \frac{g(z_0)}{z-z_0} + g'(z_0) + \cdots}$

$$f = \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$\frac{1}{z-0}\underbrace{\frac{1}{(z-1)(z-2)}}_{g(z)}$$

$$Res(f;0) = A = \frac{1}{2}$$

$$Res(f;1) = B = -1 \qquad \left(\frac{1}{z-1}\right)\frac{1}{z(z-2)}$$

$$Res(f;2) = C = \frac{1}{2} \qquad \frac{1}{z-2}\left(\frac{1}{z(z-1)}\right)$$

just as easy for $\frac{3z^2+4}{z(z-1)(z-2)}$

4.3: Application of contour integration to interesting integrals from real variables.

<u>table entry 4</u> (page 296)

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\; d\theta$$

where f is any rational function of $\cos(\theta)$, $\sin(\theta)$, or more generally any function $f(z, w)$ that is analytic in z and w for $|z|,|w| \le 1$, except for isolated singularities.This can be expressed as contour integral around the unit circle, and then evaluated using the Residue Theorem: If

$$z = e^{i\theta}, 0 \le \theta \le 2\pi$$

then

$$\frac{1}{z} = e^{-i\theta}$$

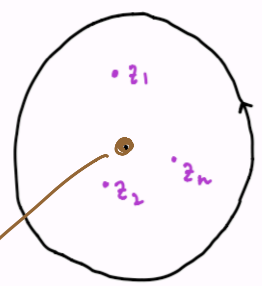
$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}.$$

$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi)$$

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$



$$= \oint_{|z|=1} f\left(\tfrac{1}{2}(z+\tfrac{1}{z}), \tfrac{1}{2i}(z-\tfrac{1}{z})\right)\frac{dz}{iz}$$

$$= 2\pi i\left(\sum_j res(\quad ; z_j)\right)$$

$|z_j| < 1$

<u>Example</u>, using an integral you probably already know, (since $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$):

$$\int_0^{2\pi} \cos^2\theta\; d\theta \quad = \pi$$

$z = 0$
only sing

Solution:

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right) \Rightarrow \cos^2(\theta) = \frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)$$

$$\int_0^{2\pi} \cos^2\theta\; d\theta = \int_{|z|=1} \frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)\frac{dz}{iz}$$

$$= 2\pi i\, Res\left(\underbrace{\frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)\left(\frac{1}{iz}\right)}_{\frac{1}{4}\,\frac{2}{i}\,\frac{1}{2}}; 0\right) = \frac{2\pi i}{2i} = \pi.$$

$Res = \frac{1}{2i}$

Example Show

$$\frac{1}{2}\int_0^{2\pi} \cos^4\theta \, d\theta = \int_0^{\pi} \cos^4\theta \; d\theta = \frac{3}{8}\pi .$$

$\cos^4\theta$ has period π

$z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$dz = e^{i\theta} i \, d\theta$

$\frac{dz}{iz} = d\theta$

$$\frac{1}{2}\oint_{|z|=1} \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^4 \frac{dz}{iz} = \frac{1}{2}\oint_{|z|=1} \frac{1}{2^4}\left[z^4 + 4\underbrace{z^3\left(\frac{1}{z}\right)}_{z^2} + 6\underbrace{z^2\frac{1}{z^2}}_{1} + 4\underbrace{z\frac{1}{z^3}}_{\frac{1}{z^2}} + \frac{1}{z^4}\right]\frac{dz}{iz}$$

only sing in unit disk @ $z=0$

$$= \frac{1}{32} 2\pi i \left(\text{Res} \,—\, ; 0\right)$$

only $\frac{1}{z}$ term contributes

$$= \frac{1}{32} 2\pi i \left(\frac{6}{i}\right) = \frac{3}{8}\pi \quad \blacksquare$$

on Friday

table entries 1 and 2 integrals of rational functions (or suitable analytic functions) over the real line. To compute

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$$

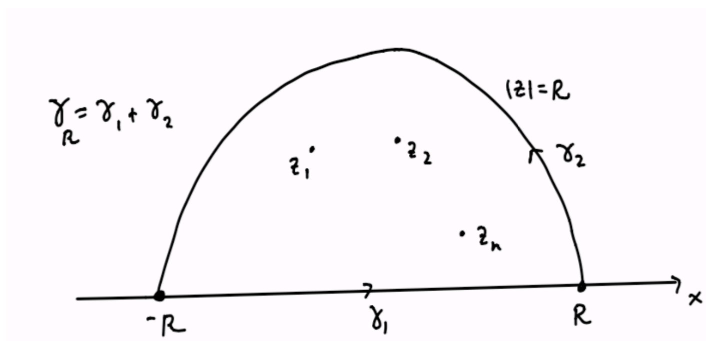
where $f(x)$ is the restriction to the real line of function $f(z)$ which is analytic on all of $\mathbb{C}$ except for a finite number of isolated singularities, none of which occur on the real line. And provided that for large $|z|$ there is a uniform modulus bound

$$|f(z)| \leq \frac{M}{|z|^2}.$$

Example (one you know) Use contour integration to show

$$\int_0^{\infty} \frac{1}{1+x^2}\, \mathrm{d}x = \frac{\pi}{2}.$$

Hint: Consider $\gamma_R = \gamma_1 + \gamma_2$, apply the Residue Theorem, and let $R \to \infty$. Make good estimates. Either choice of contour (upper semi-circle, or lower semi-circle) can potentially work for this sort of problem.

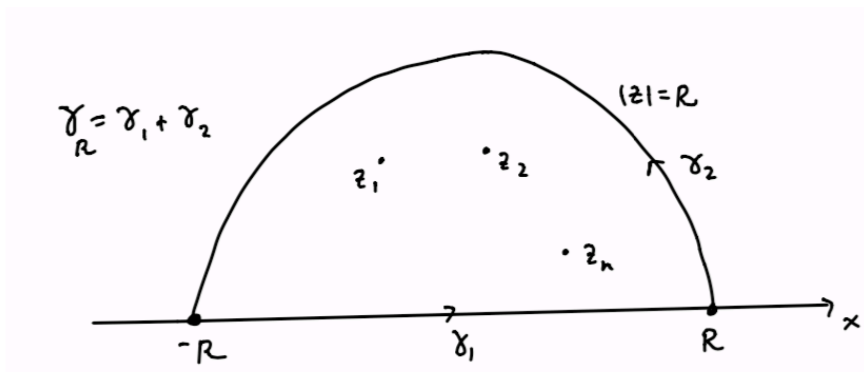


Show that if $f(x)$ is the restriction to the real line of function $f(z)$ which is analytic on all of $\mathbb{C}$ except for a finite number of isolated singularities, none of which occur on the real line; and if for large $|z|$ there is a uniform modulus bound

$$|f(z)| \leq \frac{M}{|z|^2};$$

then

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \{\text{residues of } f \text{ in the upper half plane}\}$$



Review for exam on Wednesday, which will cover 2.4-2.4, 3.1-3.3, 4.1-4.2, and implicitly use the earlier course material.

Exam begins at 11:45 and ends at 12:45.
You'll be given the text residue table, but otherwise the exam is closed book and closed note.

most important power series $\frac{1}{1-z}$

As with the first exam you'll be asked to complete 3 substantial problems, out of a choice of 5 or 6.
Additionally there may be a few required questions at the start of the exam, as was the case for Exam 1.
There will be a mixture of theorem proofs/explanations, along with computations.

basis for understanding Laurent series.

I'll go over the 2011 exam in this afternoon's problem session.

most important contour integral: $\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i\, I(\gamma;z_0)$

Topics:

2.4 Cauchy integral formula

Index $I(\gamma; z_0)$ $\quad \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\, I(\gamma;z_0)$ $\quad$ hypotheses: f anal. in A, γ closed contour homotopic to pt in A

C.I.F. for closed contour γ contractible in a domain on which $f(z)$ is analytic.

Cauchy's Theorem for domains with holes, that we proved in section 2.3

formulas and estimates for derivatives

Liouville's Theorem

Fundamental Theorem of Algebra

Morera's Theorem.

2.5 Maximum modulus principle and harmonic functions

Mean value property for $f(z)$ analytic $\quad f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$ $\quad$ if $\overline{D(z_0;r)} \subset A$, $f: A \to \mathbb{C}$ analytic.

✗ Clever proof for harmonic conjugates in simply connected domains

Mean value property for harmonic functions $\quad$ for $u(x_0, y_0)$

Maximum modulus principle for $f(z)$ analytic ✓

Maximum and minimum principles for harmonic functions ✓

3.1 Convergent sequences and series of analytic functions

why uniform limits of analytic functions are analytic, and why the derivative of the limit is the limit of the derivatives

Weierstrass M test

3.2 Power series and Taylor's Theorem

radius of convergence

term by term differentiation

uniqueness

analytic if and only if power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=k}^{\infty} a_n (z-z_0)^n = (z-z_0)^k g(z)$$

$a_k \neq 0$, $g(z_0) = a_k \neq 0$

isolated zeroes theorem → if $f(z_0)=0$ Then unless $f \equiv 0$ in $D(z_0;r)$ $\exists \rho > 0$ s.t. $f(z) \neq 0$ $\forall\ 0 < |z-z_0| < \rho$.

multiplication of power series I like these

key examples $\begin{cases} \frac{1}{1-z} \\ e^z \\ \cos z \quad \sin z \end{cases}$ compositions etc.

3.3 Laurent series

analytic in an annulus (including punctered disk case) if and only if Laurent series.....where does each piece converge?

$\rho \leq |z-z_0| \leq R$ $\quad \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$

$|z-z_0| < R$ $\quad \mathbb{C} \setminus \overline{D(z_0;\rho)}$

uniqueness

isolated zeroes classification

residue

multiplication of Laurent series.

geometric series wizardry.

4.1 Calculating residues at isolated singularities

$$f(z) = \frac{f_1(z)}{(z - z_0)^k} + f_2(z)$$

$$f(z) = \frac{g(z)}{h(z)} = \frac{\sum_{n=M}^{\infty} a_n (z - z_0)^n}{\sum_{n=N}^{\infty} \tilde{a}_n (z - z_0)^n}$$

simple poles

table if desperate.

4.2 Residue theorem

simple poles are our favorite!

statement and proof for γ contractible in A via the deformation theorem

statement and proof if γ is a simple closed curve bounding a domain via Cauchy's Theorem for domains with holes.

contour integral computations via the residue theorems

residues at ∞ (I'll remind you of the formula if you have to use it.)