Math 4200
Friday November 1
3.3 Laurent series.

Announcements:

<u>Laurent Series Theorem</u> For $0 \le R_1 < R_2$ let

$$A = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$$

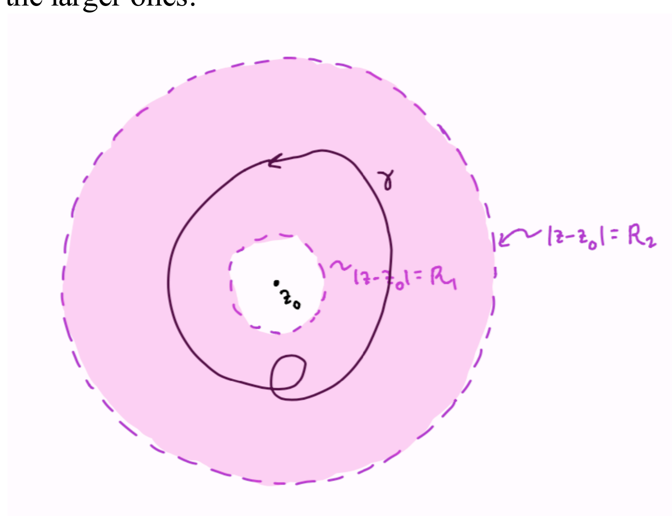
be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f : A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}.$$
$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$.
And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \ge r_1 > R_1$.
Notes: (2) $\Rightarrow$ (1) is immediate from the uniform convergence of $S_1(z) + S_2(z)$ on all compact subannuli $r_1 \le |z - z_0| \le r_2$ with $R_1 < r_1 < r_2 < R_2$. And, the uniform absolute convergence on the restricted domains follows from the convergence on the larger ones.



(3) The Laurent coefficients a_n, b_m are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_{\gamma} f(\zeta)(\zeta - z_0)^{m-1} d\zeta.$$

In particular the contour integral of f itself has value

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i\, b_1.$$

For this reason, the coefficient b_1 of $\frac{1}{z - z_0}$ in the Laurent series, is called the *residue* of f at z_0.

proof of (2) $\Rightarrow$ *(3)* in the Laurent series theorem:

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z-z_0)$:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}$$
$$:=S_1(z)+S_2(z).$$

Here $S_1(z)$ converges for $|z-z_0|<R_2$ and uniformly absolutely for $|z-z_0|\le r_2<R_2$.
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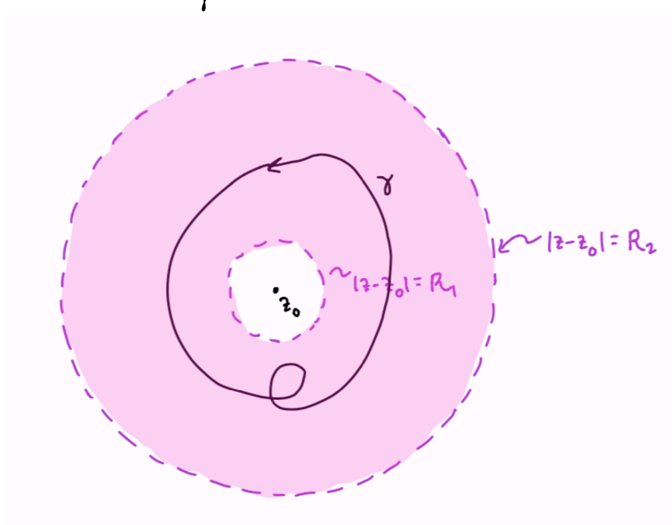
$$b_m=\frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta-z_0)^{m-1}d\zeta.$$

proof: We'll write $f(\zeta)=S_1(\zeta)+S_2(\zeta)$ and just compute the prescribed contour integrals of f, to see how they pick off the individual Laurent coefficients . We'll use the uniform convergence of the series $S_1(\zeta), S_2(\zeta)$ on γ to interchange the integrals with the summations. This is exactly the same philosophy and geometric series ideas as in the examples on Wednesday, but applied in this general context.

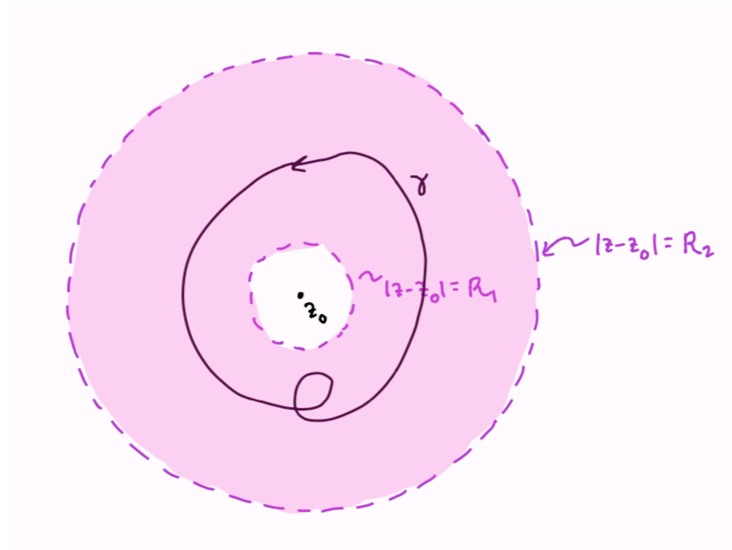
$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k+\sum_{k=1}^{\infty}\frac{b_k}{(z-z_0)^k}$$

$$a_n=\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\ ?\qquad b_m=\frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta-z_0)^{m-1}d\zeta\ ?$$



$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k}$$

$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad ? \qquad\qquad b_m = \frac{1}{2\pi i}\int_{\gamma} f(\zeta)(\zeta - z_0)^{m-1} d\zeta \quad ?$$



proof of (1) ⇒ (2) in the Laurent series theorem:

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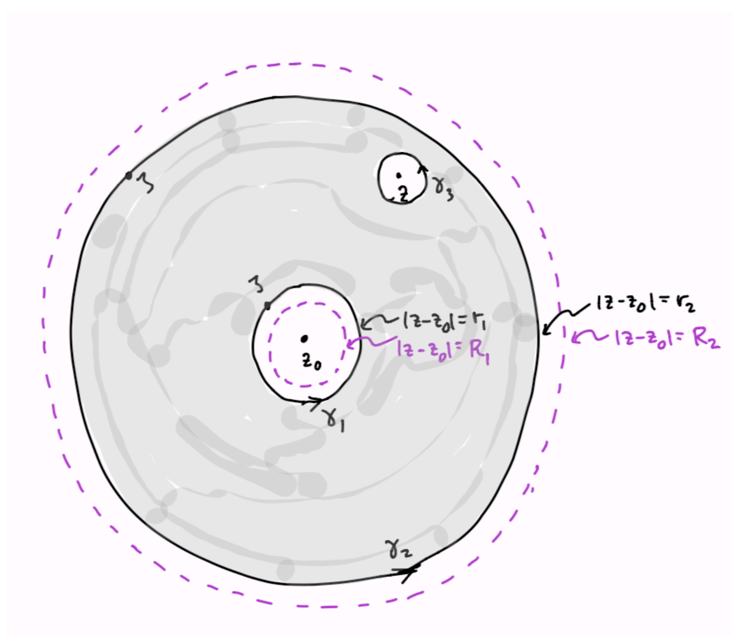
$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for and compact subdisk $|z - z_0| \le r_2 < R_2$.

And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly absolutely for any complement of a strictly larger disk, $|z - z_0| \ge r_1 > R_1$.

proof: We'll just focus on the convergence statements, because the absolute convergence statements follow from those. Let z be in the open annulus A. Pick r_1, r_2, ε so that $R_1 < r_1 < r_2 < R_2$ and so that all points of $\bar{D}(z_0; \varepsilon)$ lie in the sub-annulus $r_1 < |z - z_0| < r_2$. See figure. Let γ_1 be the circle of radius r_1 about z_0; let γ_2 be the circle of radius r_2 about z_0; let γ_3 be the circle of radius ε about z. All circles oriented counterclockwise as usual.



Then by the Green's Theorem version of Cauchy's Theorem (for domains with holes),

$$\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z} d\zeta \quad .$$

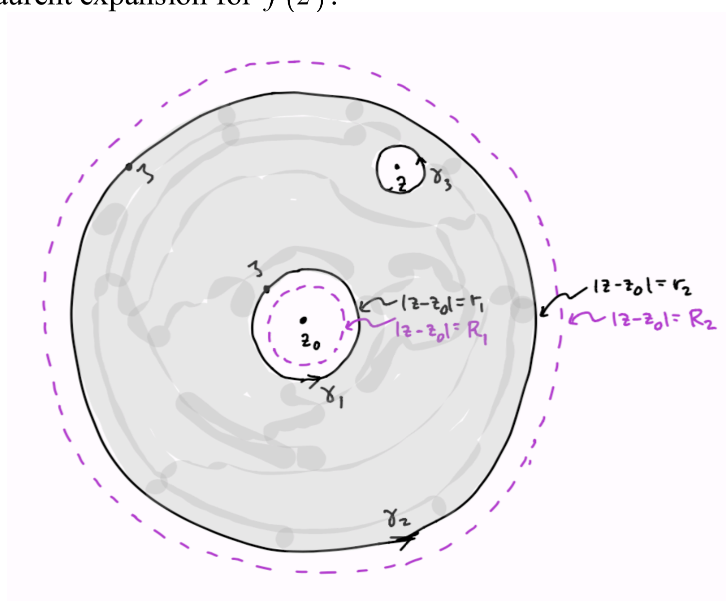
By C.I.F. on the little disk bounded by γ_3,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z} d\zeta ,$$

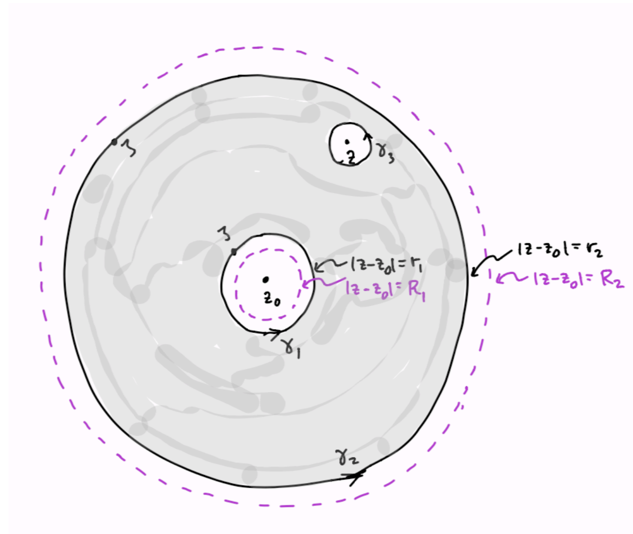
and substituting this into the formula above yields

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Use our geometric series wizardry to find the Laurent expansion for $f(z)$.



$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$



Isolated singularities table.

Let f be analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$.

type of singularity at z_0	Laurent series definition	characterization in terms of $\lim_{z\to z_0} f(z)$
removable (because f extends to be analytic at z_0)	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in L.S.)	any of: ① $\lim_{z\to z_0} f(z)=L\in\mathbb{C}$ exists ② $\lvert f(z)\rvert\le M$ $\forall\ 0<\lvert z-z_0\rvert\le\rho$ for some $0<\rho<r$. ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$.
pole (North pole!) of order N; simple pole if $N=1$	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{N}\frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$	① $\lim_{z\to z_0} f(z)=\infty$ (the north pole on the Riemann sphere) or ② $\exists\ N$ s.t. $g(z)=(z-z_0)^N f(z)$ has a removable singularity @ $z=z_0$, with $g(z_0)\neq 0$.
essential singularity	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}$ with $\{m_j\}\to\infty$, $b_{m_j}\neq 0$	(Monday) $\forall\ 0<\rho<r$: $\overline{f(D(z_0;\rho)\setminus\{z_0\})}=\mathbb{C}$! (In fact, more is true and is called "Picard's Theorem: $f(D(z_0;\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$ except for at most a single point!) e.g. $f(z)=e^{1/z}$ @ $z_0=0$: $f(D(0;\rho)\setminus\{0\})=\mathbb{C}\setminus\{1\}$ $\forall\rho>0$