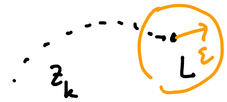
Sequences



$\epsilon - \delta$ definitions:

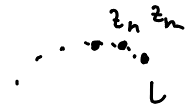
$\{z_k\} \to L$ iff $\quad \forall \varepsilon > 0 \ \exists N$ s.t. $k \geq N \Longrightarrow |z_k - L| < \varepsilon$

(note: limits are unique.)

$(k \in \mathbb{N})$

$\{z_k\}$ is *"Cauchy"* iff $\quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$. s.t. $\forall m, n \geq N$, $|z_m - z_n| < \varepsilon$

to be continued ...



Theorem The sequence $\{z_k\}$ is Cauchy iff $\{z_k\}$ converges to some limit L. in $\mathbb{C}$, or $\mathbb{R}^n$.

- $\{z_k\} \to L$. Then $\{z_k\}$ is Cauchy. Let $\varepsilon > 0$

crux $\quad |z_m - z_n| = |(z_m - L) + (L - z_n)|$
$\qquad \leq |z_m - L| + |z_n - L|$

So $\exists N$ s.t. $k \geq N \Rightarrow |z_k - L| < \varepsilon/2$

so if $m, n \geq N \Longrightarrow |z_m - z_n| < \varepsilon \Rightarrow \{z_k\}$ is Cauchy.

Theorem If $\{z_k\} \to L$ and $\{w_k\} \to M$ and $a \in \mathbb{C}$ then

(i) $\{a\, z_k\} \to a\, L$

(ii) $\{z_k + w_k\} \to L + M$

crux $\quad |z_k + w_k - (L + M)| = |(z_k - L) + (w_k - M)|$
$\qquad \leq |z_k - L| + |w_k - M|$

(iii) $\{z_k\, w_k\} \to L\, M$

crux $\quad |z_k w_k - LM| = |z_k(w_k - L) + L(w_k - M)|$
$\qquad \leq |z_k||w_k - L| + |L||w_k - M|$

(iv) $\left\{\dfrac{z_k}{w_k}\right\} \to \dfrac{L}{M}$ provided $w_k \neq 0 \ \forall\, k$ and $M \neq 0$.

use (iii) & special case $\left\{\dfrac{1}{w_k}\right\} \to \dfrac{1}{M}$

- Let $\{z_k\}$ Cauchy.

Let $\varepsilon > 0$ $\quad$ Crux $\quad k, \ell \geq N$

$|z_k - z_\ell| < \varepsilon$

$\begin{cases} z_k = x_k + i y_k \\ z_\ell = x_\ell + i y_\ell \end{cases}$

$|x_k - x_\ell|, |y_k - y_\ell| \leq |z_k - z_\ell| = \sqrt{(x_\ell - x_k)^2 + (y_\ell - y_k)^2}$

so $\{x_k\}, \{y_k\} < \mathbb{R}$ are Cauchy.

$\mathbb{R}$ complete,

so $\{x_k\} \to x$
$\{y_k\} \to y$

follows that
$\{z_k\} \to z = x + iy$.

sequences and sets

Def Let $B \subseteq \mathbb{C}$. $z \in \mathbb{C}$ is a *"limit point"* of B iff $\exists \{z_k\} \subseteq B$ s.t. $z_k \to z$.

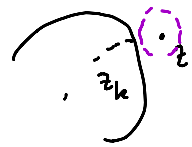
* Theorem $B \subseteq \mathbb{C}$ is closed iff B contains all of its limit points.

- Let B be closed. Let $\{z_k\} \subset B$, $\{z_k \to z\}$ show $z \in B$.

If $z \notin B$ then $z \in B^c$ open

$\uparrow$ complement.

$\Rightarrow \exists D(z;\varepsilon) \subset B^c$ $(\varepsilon > 0)$.



$\Rightarrow\Leftarrow$ (Pick N s.t. $k \geq N \Rightarrow z_k \in D(z;\varepsilon) \cap B$. There is no such z_k).

- Let B contain all its limit points. Show B is closed. Show B^c is open.

Let $z \in B^c$ need $D(z;\varepsilon) \subset B^c$ contains

If no such ε, then each $D(z, \frac{1}{n}) \ni z_n \in B. \Rightarrow \{z_n\} \to z$

$\Rightarrow z \in B$ $\Rightarrow\Leftarrow$

Def Let $B \subseteq \mathbb{C}$. The *"closure of B"* is the union of B with all of its limit points.

Theorem Let $B \subseteq \mathbb{C}$. Then the closure of B is closed, and can also be characterized as the intersection of all closed sets containing B.

* Theorem The following are equivalent for $K \subseteq \mathbb{C}$. True in $\mathbb{R}^n$

(i) K is compact. every open cover has finite subcover.

$\Downarrow$

(ii) K is closed and bounded.

$\Downarrow$

(iii) Every sequence $\{z_k\} \subseteq K$ has a convergent subsequence $\{z_{k_j}\}$ with $\{z_{k_j}\} \to z_0 \in K$.

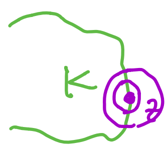
(iii) $\Rightarrow$ (i): following page

proof: $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are reasonable to reconstruct in class. $(iii) \Rightarrow (i)$ is somewhat more intricate, so I'll include a proof on the next page which you can read or recall at your leisure, if you wish.

$(i) \Rightarrow (ii)$

- If K is compact, it's bounded. Cover K by $\bigcup_{n \in \mathbb{N}} D(0;n)$ open cover. There's a finite subcover $K \subset \bigcup D(0;n_j)$, $n_1 < n_2 < \dots N$, $K \subset D(0,N)$.
- If K is compact, it's closed

If K is not closed it has a limit point z it doesn't contain



consider open cover $K \subset \bigcup_n \underbrace{\overline{\underbrace{D(z;\tfrac{1}{n})}_{\text{closed disk}}}^{C}}_{=\mathbb{C} \setminus \{z\}}$ complements.

But there's a finite subcover, increasing union

so $K \subset \overline{D(z;\tfrac{1}{N})}^{C}$ some N.

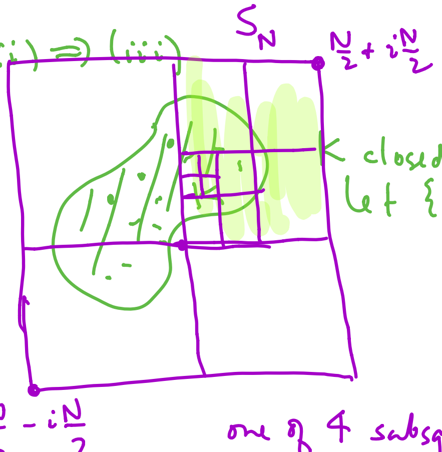
i.e. $K \cap D(z;\tfrac{1}{N}) = \emptyset$

$\Rightarrow\!\Leftarrow$ since z is a limit point of K

so $\exists \{z_k\} \subset K$, $\{z_k\} \to z$

$\blacksquare$

$(ii) \Rightarrow (iii)$



K closed & bd.

Let $\{z_k\} \subset K$ seq.

one of 4 subsquares, $S_{N/2}$ has ∞'ly many $\{z_{k_j}\}$

one of its subsquares has ∞'ly $S_{N/4}$

$\{z_{k_{j_\ell}}\}$

so construct Cauchy subseq.

$\{z_1, z_{k_1}, z_{k_{j_1}}, \dots\} \to z_0$. Cauchy seqs conv.

And since z_0 is a limit point of closed K, it's in K.

$\blacksquare$

I forgot to say this in class

$(iii) \Rightarrow (i)$. We assume $K \subseteq \mathbb{C}$ has the property that every sequence $\{z_k\} \subseteq K$ has a convergent subsequence $\{z_{k_j}\}$ with $\{z_{k_j}\} \to z_0 \in K$. We wish to show that every open cover of K has a finite subcover. We will use the fact that the rational points $\{p + i\,q \mid p, q \in \mathbb{Q}\}$ are dense in $\mathbb{C}$, specifically the fact that $\mathbb{C}$ is separable:

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of K. For each $z \in U_\alpha$ pick $z_p, r_p \in \mathbb{Q}$ s.t $z \in \mathrm{D}(z_p, r_p) \subseteq U_\alpha$. Then the collection $\{\mathrm{D}(z_p, r_p)\}_{z \in U_\alpha, \alpha \in A}$ is a highly redundant countable cover of K. Re-label the non-redundant disks by the natural numbers,

$$\{\mathrm{D}(z_k, r_k)\}_{k \in \mathbb{N}}.$$

If we can find a finite subcover of K using these disks, then since each disk $\mathrm{D}(z_k, r_k)$ is in some U_{α_k}, the finite collection $\{U_{\alpha_k}\}$ will also be a finite subcover of K, and the theorem will be proven. We prove this fact by contradiction:

If no finite subcover exists, then for each $n \in \mathbb{N}$ pick

$$w_n \in K, w_n \notin \cup_{k=1}^{n} \mathrm{D}(z_k, r_k).$$

By the assumption (iii) a subsequence $\{w_{n_j}\} \to w_0 \in K$. But this $w_0 \in \mathrm{D}(z_k, r_k)$ for some fixed k since the collection of all these disks is a cover of K. Thus by convergence, there exists J s.t. $j \geq J \Rightarrow w_{n_j} \in \mathrm{D}(z_k, r_k)$. This violates how w_{n_j} was chosen, as soon as $n_j > k$. This contradiction proves that for some N,
$K \subseteq \cup_{k=1}^{N} \mathrm{D}(z_k, r_k) \subseteq \cup_{k=1}^{N} U_{\alpha_k}$.

Math 4200-001

Friday August 30

Part 2 of section 1.4: Add functions to the mix of sets, sequences, in review of 3220 material we'll be using in 4200.

Announcements

- hw 1 solutions are posted, but hw isn't graded yet → will be returned on Wed.
- We'll continue our review of 3220 for 4200, trying to not rush & also to highlight key results. (Probably won't finish all of "today's" notes today.) ← we didn't even start them, but we did finish Wed. notes
- Don't come to class on Monday.

Warm-up exercise