Math 4200-001

Friday August 30
Part 2 of section 1.4: Add functions to the mix of sets, sequences, in review of 3220 material we'll be using in 4200.

Announcements

Warm-up exercise

The functions we'll be using this course will almost be one of the following three types:

i) For $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$ complex functions (or the associated $F : A \to \mathbb{R}^2$, with $A \subseteq \mathbb{R}^2$).

ii) For $I \subseteq \mathbb{R}$, $\gamma : I \to \mathbb{C}$ curves (or the associated $\Gamma : I \to \mathbb{R}^2$).

iii) For $A \subseteq \mathbb{C}$(or $A \subseteq \mathbb{R}^2$), $u : A \to \mathbb{R}$ real-valued functions.

Since continuity is easily discussed for $F : A \to \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$ we will temporarily adopt that generality, and use the 3220 notation: For $x_0 \in \mathbb{R}^n$,

$$B_\varepsilon(x_0) := \left\{x \in \mathbb{R}^n \,\middle|\, \|x - x_0\| < \varepsilon\right\}$$

instead of the disk notation $D(z_0; \varepsilon)$ which is special to $\mathbb{C}$.

<u>Def</u> Let $A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$.

For $B \subseteq A$, $F(B) := \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in B\}$ "image of B"

For $C \subseteq \mathbb{R}^n$, $F^{-1}(C) := \left\{x \in A \,\middle|\, f(x) \in C\right\}$ "inverse image of C"

<u>Def</u> Let $A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$, and $x_0 \in A$. Then F *is continuous at* x_0 iff

Note: If $F : A \to \mathbb{R}^2$ and we consider the associated $f : A \to \mathbb{C}$, then f *is continous at* x_0 iff

(and this definition is equivalent to F continous at x_0)

Def Let $A \subseteq \mathbb{R}^n$ and $F: A \to \mathbb{R}^m$. Then *F is continuous on A* iff F is continuous at each $x_0 \in A$.

Def Let $A \subseteq \mathbb{R}^n$, $U \subseteq A$. Then *U is relatively open (in A)* iff $\exists\ V \subseteq \mathbb{R}^n$ open, such that $U = V \cap A$.
(Note, U is relatively open in A iff $\forall\ x \in U\ \exists\ \delta > 0$ such that $B_\delta(x) \cap A \subseteq U$.)

Let $A \subseteq \mathbb{R}^n$, $W \subseteq A$. Then *W is relatively closed (in A)* iff $\exists\ V \subseteq \mathbb{R}^n$ closed, such that $W = V \cap A$.

Theorem Let $A \subseteq \mathbb{R}^n$, $F: A \to \mathbb{R}^m$, $x_0 \in A$. The following are equivalent:

(i) F is continuous at x_0 ($\varepsilon - \delta$ definition)
(ii) F is sequentially continuous at x_0, i.e. $\forall\ \{x_n\} \subseteq A$ such that $\{x_n\} \to x_0$, then also $\{F(x_n)\} \to F(x_0)$.

(You essentially prove this theorem in your homework problem 1.4.18., so we can skip it here.)

<u>Theorem</u> Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$. The following are equivalent

(i) F is continuous on A

(ii) For all $O \subseteq \mathbb{R}^m$ open, $F^{-1}(O)$ is (relatively) open in A.

This is an important and useful characterization of continous functions, so we'll recall or construct a proof in class.

Theorem $A \subseteq \mathbb{R}^n$, $f, g : A \to \mathbb{R}$; or $F, G : A \to \mathbb{R}^2$ with associated complex functions $f, g : A \to \mathbb{C}$. Then f, g continous at $x_0 \in A$ implies

(i) $f + g$ is continous at x_0

(ii) fg is continous at x_0

(iii) $\dfrac{f}{g}$ is continous at x_0 if $g(x_0) \neq 0$.

proof: Since continuity is equivalent to sequential continuity it suffices to consider all $\{x_n\} \subseteq A$ with $\{x_n\} \to x_0$, and assuming $\{f(x_n)\} \to f(x_0)$, $\{g(x_n)| \to g(x_0)$ and to then show that

$$\{f(x_n) + g(x_n)\} \to f(x_0) + g(x_0)$$

$$\{f(x_n)g(x_n)\} \to f(x_0)g(x_0)$$

$$\left\{\frac{f(x_n)}{g(x_n)}\right\} \to \frac{f(x_0)}{g(x_0)}.$$

But this follows from the analogous Theorem for sequences in Wednesday's notes. (You could also prove this theorem with $\varepsilon's$ and $\delta's$.)

Continuous functions and compactness:

(1) Theorem Let $K \subseteq \mathbb{R}^n$ compact, $F : K \to \mathbb{R}^n$ continuous. Then $F(K) \subseteq \mathbb{R}^m$ is compact.
proof:

(2) Corollary (Extreme value Theorem) Let $K \subseteq \mathbb{R}^n$ compact, $F : K \to \mathbb{R}$ continuous. Then F attains its infimum m and supremum M. In other words,

$$\exists\, x_1 \in K \text{ such that } f(x_1) = inf_{x \in K} f(x).$$
$$\exists\, x_2 \in K \text{ such that } f(x_2) = sup_{x \in K} f(x).$$

proof: By Theorem 1, $F(K)$ is compact. Because compact sets in $\mathbb{R}^n$ are bounded the infimum and supremum are finite. Because compact sets are closed, and closed sets contain all their limit points, we know that $F(K)$ contains all its limit points. Thus $F(K)$ contains its infimum and extremum. QED.

Def Let $A \subseteq \mathbb{R}^n$, $F : A \to \mathbb{R}^m$. Then F *is uniformly continuous* iff

(3) Theorem Let $K \subseteq \mathbb{R}^n$, K compact. Let $F : K \to \mathbb{R}^m$ be continous. Then F is uniformly continuous.

proof (see text, or try to reconstruct on your own. Various approaches work after some effort.)

Connectivity and functions

The following definition is equivalent to the one we wrote down on Wednesday:

A set $A \subseteq \mathbb{R}^n$ is *"not connected"* (or *"has a disconnection"*) iff $\exists\ U, V$ relatively open subsets of A such that

$$A = U \cup V$$

$$U \neq \varnothing,\ V \neq \varnothing$$

$$U \cap V = \varnothing .$$

A set $A \subseteq \mathbb{R}^n$ is *"connected"* if it has no disconnection.

(1) Theorem Let $A \subseteq \mathbb{R}^n$ be connected, $F : A \rightarrow \mathbb{R}^m$ continuous. Then $F(A)$ is connected.
proof:

Def Let $A \subseteq \mathbb{R}^n$. A is *path connected* iff $\forall\ x, y \in A,\ \exists\ \gamma : [a, b] \rightarrow A$ continuous (a "path") such that $\gamma(a) = x, \gamma(b) = y$.

(2) Theorem If A is path connected, then A is connected.

proof: (sketch) Assume $\{U, V\}$ is a disconnection of A. Pick $x \in A$, $y \in B$ and let $\gamma : [a, b] \rightarrow A$ be a (continuous) path connecting x to y. Let

$$c := sup\{t \geq a \mid \gamma([a, t]) \subseteq A\}$$

We can show that $c > a$, $c < b$ and that $\gamma(c)$ is in neither U nor V, using the facts that U, V are both relatively open. This is a contradiction to the assumption that $\{U, V\}$ covers A. Thus, no disconnection of A exists.

(3) Theorem If A is open and connected, then A is path connected. (Thus, for open sets, being path connected is the same as being connected.)

proof (sketch). Pick any $x_0 \in A$. Let $U = \{y \in A$ such that there is a path from x_0 to $y\}$. We can show that U is open, and that its complement $V := A \setminus U$ is also open, unless it is empty. Thus V is empty, since A is connected.