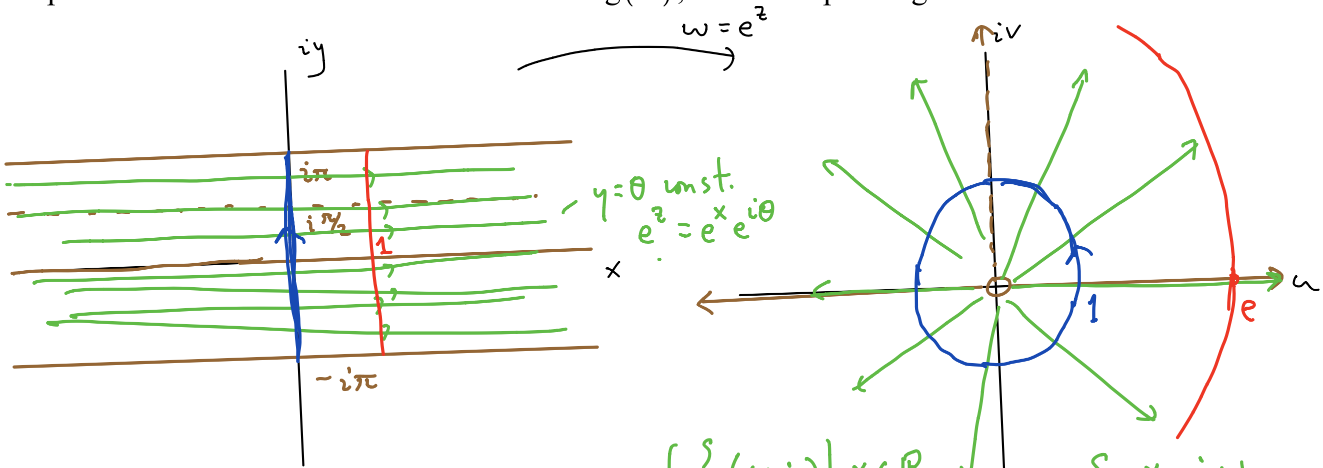
vertical lines or segments
e.g. $\{x_0 + iy \mid -\pi < y < \pi\} \to \{e^{x_0} e^{iy} \mid -\pi < y < \pi\}$
↗ circles of radius e^{x_0}

<u>Example 4</u> For $z = x + iy$, $x, y \in \mathbb{R}$

$$f(z) = e^z = e^{x+iy} := e^x e^{iy}$$

Discuss and sketch how f transforms the z-plane into an infinitely covered w-plane. For $w = e^z$ discuss possible continuous inverse functions $z = \log(w)$, and corresponding domains.



$f\{(x+iy) \mid x \in \mathbb{R}, -\pi \le y \le \pi\} = \{e^x e^{iy} \mid x \in \mathbb{R}, -\pi \le y \le \pi\}$

is all of w-plane

any such strip (horiz) of width 2π has image covering $\mathbb{C}$

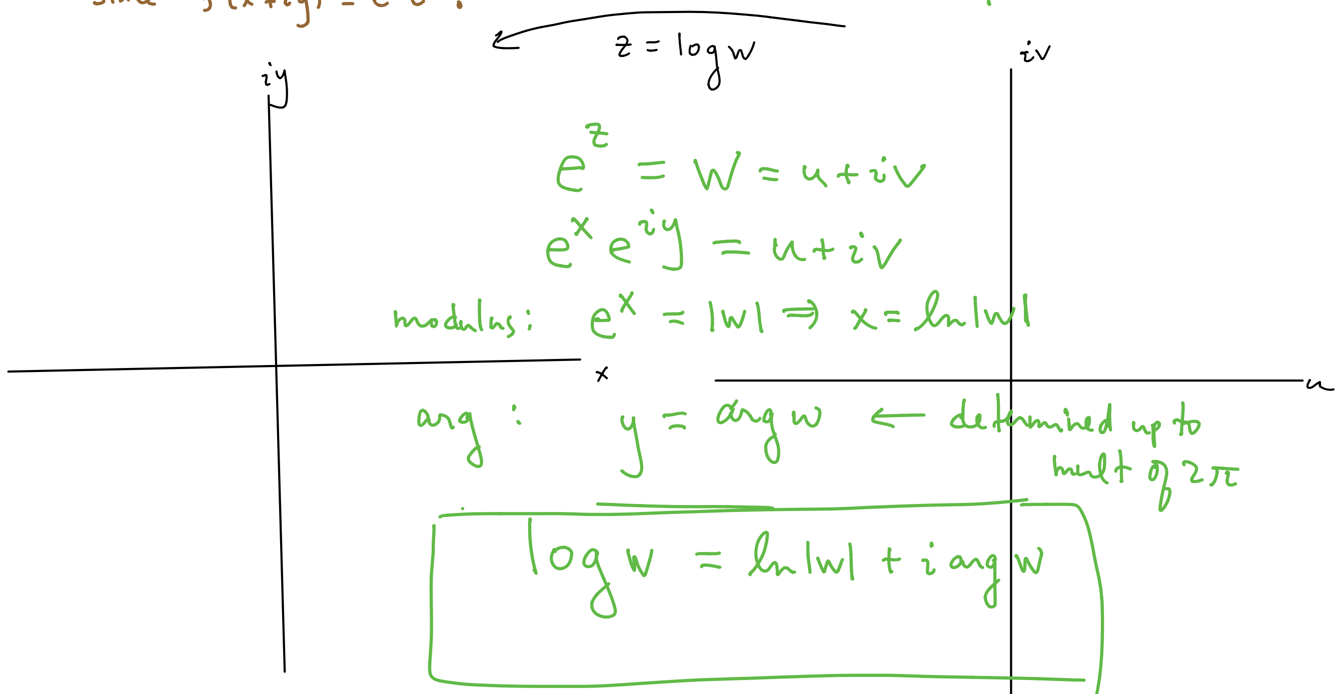
e^z is periodic in y-dir. period 2π

$f(x\text{-axis}) = +\ x\text{-axis}$

$f\{x + i\pi, x \in \mathbb{R}\} = \{e^x \underset{-1}{\underset{\|}{e^{i\pi}}}, x \in \mathbb{R}\} = -\ x\text{-axis}$

$f\{x + \frac{i\pi}{2}, x \in \mathbb{R}\} = \{e^x \underbrace{e^{i\pi/2}}_{i}, x \in \mathbb{R}\} = +\ i$ axis

if horiz line at iy
its image is a ray with argument y
since $f(x+iy) = e^x e^{iy}$.



$$e^z = w = u + iv$$

$$e^x e^{iy} = u + iv$$

modulus: $e^x = |w| \Rightarrow x = \ln|w|$

arg: $y = \arg w$ ← determined up to mult of 2π

$$\boxed{\log w = \ln|w| + i \arg w}$$

to be continued!

Example 5 "Trig functions".
If x is real,

$$\text{eqtn 1} \qquad e^{ix} = \cos(x) + i\sin(x)$$
$$\text{eqtn 2} \qquad e^{-ix} = \cos(x) - i\sin(x).$$

$e^{i(-x)} := \cos(-x) + i\sin(-x)$
$= \cos x - i \sin x$

$\dfrac{eqtn\ 1 + eqtn2}{2} \Rightarrow$

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$\dfrac{eqtn\ 1 - eqtn2}{2\,i} \Rightarrow$

$$\sin(x) = \frac{1}{2\,i}\left(e^{ix} - e^{-ix}\right).$$

Also recall the hyperbolic trig functions

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$
$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$

curiosity in Calc.
showed up DE's.

So we define, for $z \in \mathbb{C}$,

$$\begin{cases} \cos(z) := \dfrac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin(z) = \dfrac{1}{2\,i}\left(e^{iz} - e^{-iz}\right) \end{cases} \qquad \begin{aligned} \cosh(z) &:= \frac{1}{2}\left(e^{z} + e^{-z}\right) = \cos(iz) = \frac{1}{2}\left(e^{iiz} + e^{-iiz}\right) \\ & = \frac{1}{2}\left(e^{-z} + e^{z}\right) \checkmark \\ \sinh(z) &:= \frac{1}{2}\left(e^{z} - e^{-z}\right) = -i\sin(iz). \end{aligned}$$

"Trig" identities hold, via properties of complex exponential multiplication. (but note that $\sin(z)$, $\cos(z)$ are no longer bounded functions....it's challenging to figure out their transformation pictures like we did for the earlier examples).

$e^{z} e^{w} = e^{z+w}$
any $z, w \in \mathbb{C}$
(warmup lemma)

$$\sin^2 z + \cos^2 z = 1$$

$$\left(\frac{1}{2i}\right)^2\left(e^{iz} - e^{-iz}\right)^2 + \frac{1}{4}\left(e^{iz} + e^{-iz}\right)^2$$
$$-\frac{1}{4}\left[\left(e^{iz}\right)^2 - e^{iz}e^{-iz} - e^{-iz}e^{iz} + e^{-iz}e^{-iz}\right] + \frac{1}{4}\left[e^{i2z} + 2 + e^{i(-2z)}\right]$$

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

$$= -\frac{1}{4}\left[\cancel{e^{i2z}} - 2 + \cancel{e^{i(-2z)}}\right]$$
$$+ \frac{1}{4}\left[\cancel{e^{i2z}} + 2 + \cancel{e^{i(-2z)}}\right]$$

$$\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \checkmark$$

trigh ...

$$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) + \sin^2(iz) = 1 \quad .....$$

Math 4200-001
Week 2 concepts and homework
1.3-1.4
Due Wednesday September 4 at start of class.

1.3 The first four problems were postponed from the first assignment: 4b, 7a, 8a, 30b. Also, add new problems 25, 31.

1.4 1, 2, 3, 4, 5, 8, 9, 11, 14 (also sketch the sets) 16 (also sketch the sets),18. (18 is the theorem that continuity is equivalent to sequential continuity.) 20, 21.

w2.1
a) Prove that the set in 14b is closed, by showing that its complement is open (directly from the definition of open sets).
b) Prove that the same set is closed, using the generalization of Proposition 1.4.9 which is true for maps from $\mathbb{C}$ to $\mathbb{R}$ or $\mathbb{R}^m$ as well.

w2.2 Prove that if $K \subseteq \mathbb{C}$ is compact, and if $K \subseteq O$, where O is open, then there exists an $\varepsilon > 0$ such that for each $z \in K$, $D(z; \varepsilon) \subseteq O$. (This is equivalent to Distance Lemma 1.4.21 in the text. See if you can find a proof without looking there first, but in any case write a proof in your own words.)

Math 4200-001

Wednesday August 28 Finish discussing section 1.3, complex transformations; begin section 1.4. We'll spend at least two days in section 1.4, which is a review of key analysis facts that we'll need in this course and that (I think) you've seen in Math 3220 earlier.

Announcements On homework: I'll post solutions to all problems, on Canvas but will only grade a subset; hope to return on Fridays.

Warm-up exercise Recall for $z = x+iy$, $e^z := e^{x+iy} := e^x e^{iy}$ $(= e^x(\cos y + i\sin y))$

~~Is it~~ It is true that $e^{z+w} = e^z e^w$? $(w = u+iv \in \mathbb{C})$

$$e^{z+w} = e^{x+iy+u+iv} = e^{(x+u)+i(y+v)} := e^{x+u}\underbrace{e^{i(y+v)}}_{=e^{iy}e^{iv}}$$

(trig identities) ✓

$$= e^x e^u e^{iy} e^{iv}$$

$$= \underbrace{e^x e^{iy}}_{e^z}\ \underbrace{e^u e^{iv}}_{e^w}$$

Section 1.3 power functions.

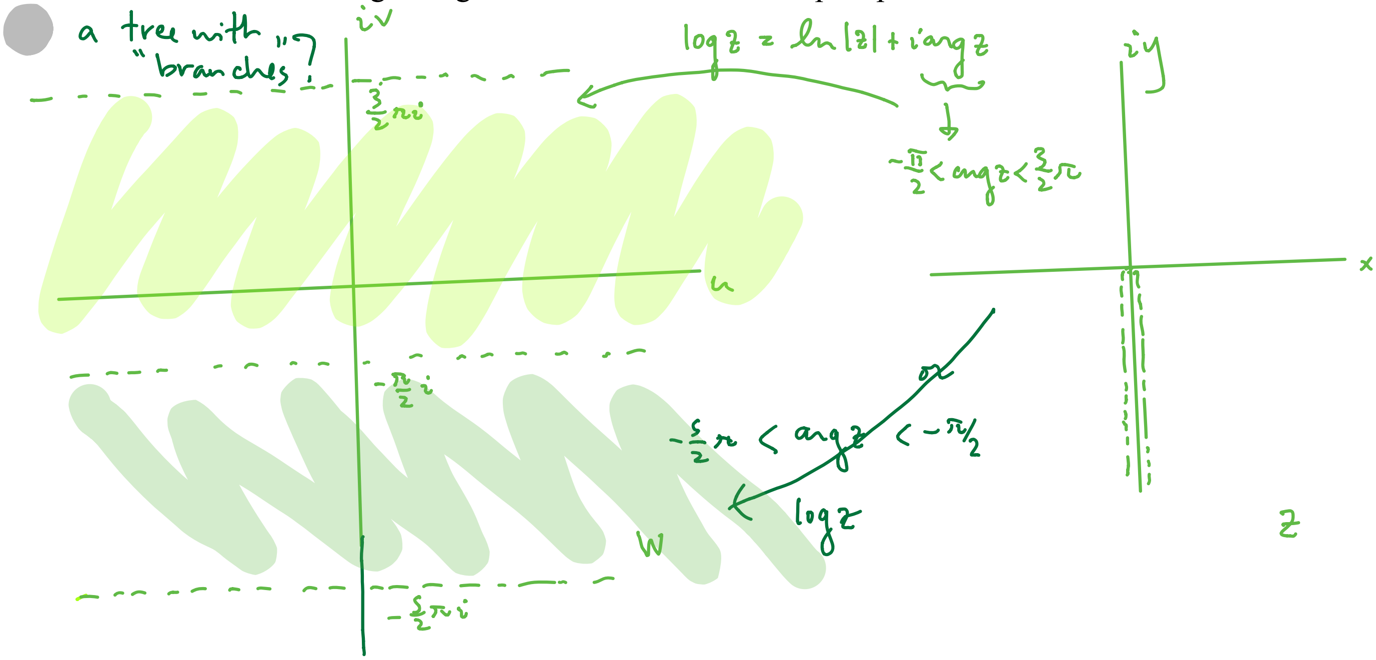
Recall

$$\log z = \ln|z| + i\,arg(z)\,.$$

$\downarrow$ text they write $\log = \log_e$

Since $arg(z)$ is only determined up to an integer multiple of 2π we always assume we're working in a domain for which $arg(z)$ has been chosen so that it's a continuous function. Typically but not always these domains can be slit domains $\mathbb{C}\setminus\gamma$, where γ is a ray starting at the origin. Each such choice for the $arg(z)$ function gives us what is called a "branch" of the $\log(z)$ function. In the context of the slit domains the image of given choice $\log(z)$ will be a horizontal strip of the complex plane, and the images of these different branches glue together to make an entire complex plane.

a tree with "branches"?



Notice that for any branch of the logarithm,

$$e^{\log z} = z:$$

$$e^{\ln(|z|) + i\cdot(arg(z) + 2\pi k)} = e^{\ln(|z|)}e^{i\,arg(z)}e^{i\,2\pi k} = |z|\,e^{i\,arg(z)} = z\,.$$

In fact,

1) If $n \in \mathbb{Z}$, then

$$z^n = e^{n\log(z)}$$

regardless of branch choice for logarithm.

$$\Downarrow$$

$$\overset{..}{\text{!!}}$$

$$e^{n(\ln|z| + i\arg z + i2\pi k)}$$

$$= \underbrace{\underbrace{e^{n\ln|z|}\; e^{in\overset{\theta}{\arg z}}}_{|z|^n e^{i(n\theta)}}}_{z^n}\;\underbrace{e^{i(2\pi kn)}}_{1}$$

✓

2) If $q = \frac{m}{n}$ is a rational number in lowest terms (m, n no common divisors, $n > 0$), then the m^{th} powers of the n^{th} roots of z,

$$z^{\frac{m}{n}}$$

are also recovered from the formula

$$e^{\frac{m}{n}\log(z)}.$$

3) So, for general $w \in \mathbb{C}$ and a branch choice of $\log(z)$, we define

$$z^w := e^{w\log(z)}.$$

This is consistent with our previous power definitions for rational numbers, but in the general case each branch choice for $\log(z)$ leads to a different branch of z^w, unlike in cases (1),(2).

Section 1.4: Analysis related to sets, distance, topology and functions, that we'll need for this course. We'll focus on sets and sequences today, and on Friday we'll focus on functions. For most of the time we'll focus on subsets of $\mathbb{C}$ and measure distances equivalently to how we do in $\mathbb{R}^2$. Sometimes we'll generalize to $\mathbb{R}^n$ discussions and more general concept definitions from Math 3220.

As we've already mentioned, for $z = x + i\,y$, $w = u + i\,v$ with $x, y, u, v \in \mathbb{R}$, we measure the distance in $\mathbb{C}$ from z to w by

$$dist(z, w) := |z - w| = \sqrt{(x-u)^2 + (y-v)^2} = \left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|$$

which is the Euclidean distance between the corresponding points in $\mathbb{R}^2$.

Sets: $B_r(\vec{x}_0)$ in 3220

$D(z_0; r) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$ "*open disk of radius r*" "*r-neighborhood of* z_0" $(r > 0)$

$D(z_0; r) \setminus \{z_0\} := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$ "*deleted open disk*" $(r > 0)$

$A \subseteq \mathbb{C}$ is a "*neighborhood of* z_0" if and only if $\exists\, r > 0$ such that $D(z_0; r) \subseteq A$.

$A \subseteq \mathbb{C}$ is "*open*" iff $\forall\, z_0 \in A\ \exists\, r > 0$ s.t. $D(z_0; r) \subseteq A$.

For $F \subseteq \mathbb{C}$ the "*complement of F*" is defined as $\mathbb{C} \setminus F := \{z \in \mathbb{C}$ s.t. $z \notin F\}$. F^c in 3220

$B \subseteq \mathbb{C}$ is "*closed*" iff its complement $\mathbb{C} \setminus B$ is open.

Theorem for open sets:
a) $\varnothing, \mathbb{C}$ are open;
b) The union of any collection of open sets is open;
c) The intersection of any finite collection of open sets is open.

Using DeMorgan's laws that complements of unions are intersections of complements; and that complements of intersections are unions of complements:

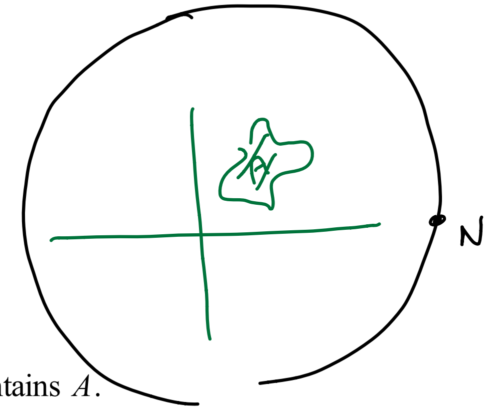
$$\mathbb{C} \setminus \left(\bigcup_{\gamma \in \Gamma} A_\gamma \right) = \bigcap_{\gamma \in \Gamma} (\mathbb{C} \setminus A_\gamma)$$

$$\mathbb{C} \setminus \left(\bigcap_{\gamma \in \Gamma} A_\gamma \right) = \bigcup_{\gamma \in \Gamma} (\mathbb{C} \setminus A_\gamma)$$

yields

Theorem for closed sets (that you've seen):
a) $\varnothing, \mathbb{C}$ are ~~open;~~ closed
b) The intersection of any collection of closed sets is closed;
c) The union of any finite collection of closed sets is closed.

<u>Some further special properties of sets</u>

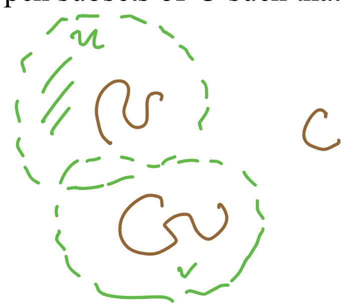


$A \subseteq \mathbb{C}$ is *"bounded"* iff $\exists\ N \in \mathbb{R}$ s.t. $|z| \leq N \quad \forall z \in A$.

An *"open cover of A"* is a collection of open sets whose union contains A.

$K \subseteq \mathbb{C}$ is *"compact"* iff every cover of K by open sets has a *"finite subcover"*, i.e. a finite subcollection of the original collection already covers K.

A set $C \subseteq \mathbb{C}$ is *"not connected"* (or *"has a disconnection"*) iff $\exists\ U, V$ open subsets of $\mathbb{C}$ such that



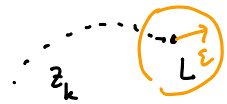
$$C \subseteq U \cup V$$

$$C \cap U \neq \varnothing, \quad C \cap V \neq \varnothing$$

$$(C \cap U) \cap (C \cap V) = \varnothing$$

all hold.

A set $C \subseteq \mathbb{C}$ is *"connected"* iff it has no disconnection.



Sequences

$\epsilon - \delta$ definitions:

$\{z_k\} \to L$ iff $\quad \forall \varepsilon > 0 \ \exists N$ s.t. $k \geq N \Longrightarrow |z_k - L| < \varepsilon$

(note: limits are unique.)

$(k \in \mathbb{N})$

$\{z_k\}$ is *"Cauchy"* iff $\quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N, \ |z_m - z_n| < \varepsilon$

to be continued ...

Theorem The sequence $\{z_k\}$ is Cauchy iff $\{z_k\}$ converges to some limit L.

Theorem If $\{z_k\} \to L$ and $\{w_k\} \to M$ and $a \in \mathbb{C}$ then

(i) $\{a\, z_k\} \to a\, L$

(ii) $\{z_k + w_k\} \to L + M$

(iii) $\{z_k\, w_k\} \to L\, M$

(iv) $\left\{ \dfrac{z_k}{w_k} \right\} \to \dfrac{L}{M}$ provided $w_k \neq 0 \ \forall\ k$ and $M \neq 0$.