Math 4200-001

Wednesday August 28 Finish discussing section 1.3, complex transformations; begin section 1.4. We'll spend at least two days in section 1.4, which is a review of key analysis facts that we'll need in this course and that (I think) you've seen in Math 3220 earlier.

Announcements

Warm-up exercise

Section 1.3 power functions.

Recall

$$\log z = \ln |z| + i\, arg(z).$$

Since $arg(z)$ is only determined up to an integer multiple of 2π we always assume we're working in a domain for which $arg(z)$ has been chosen so that it's a continuous function. Typically but not always these domains can be slit domains $\mathbb{C} \setminus \gamma$, where γ is a ray starting at the origin. Each such choice for the $arg(z)$ function gives us what is called a "branch" of the $\log(z)$ function. In the context of the slit domains the image of given choice $\log(z)$ will be a horizontal strip of the complex plane, and the images of these different branches glue together to make an entire complex plane.

Notice that for any branch of the logarithm,

$$e^{\log z} = z:$$

$$e^{\ln(|z|) + i \cdot (arg(z) + 2\pi k)} = e^{\ln(|z|)} e^{i\, arg(z)} e^{i\, 2\pi k} = |z|\, e^{i\, arg(z)} = z.$$

In fact,

1) If $n \in \mathbb{Z}$, then

$$z^n = e^{n \log(z)}$$

regardless of branch choice for logarithm.

2) If $q = \frac{m}{n}$ is a rational number in lowest terms (m, n no common divisors, $n > 0$), then the m^{th} powers of the n^{th} roots of z,

$$z^{\frac{m}{n}}$$

are also recovered from the formula

$$\mathrm{e}^{\frac{m}{n}\log(z)}.$$

3) So, for general $w \in \mathbb{C}$ and a branch choice of $\log(z)$, we define

$$z^w := \mathrm{e}^{w\log(z)}.$$

This is consistent with our previous power definitions for rational numbers, but in the general case each branch choice for $\log(z)$ leads to a different branch of z^w, unlike in cases (1),(2).

Section 1.4: Analysis related to sets, distance, topology and functions, that we'll need for this course. We'll focus on sets and sequences today, and on Friday we'll focus on functions. For most of the time we'll focus on subsets of $\mathbb{C}$ and measure distances equivalently to how we do in $\mathbb{R}^2$. Sometimes we'll generalize to $\mathbb{R}^n$ discussions and more general concept definitions from Math 3220.

As we've already mentioned, for $z = x + i\,y$, $w = u + i\,v$ with $x, y, u, v \in \mathbb{R}$, we measure the distance in $\mathbb{C}$ from z to w by

$$dist(z, w) := |z - w| = \sqrt{(x-u)^2 + (y-v)^2} = \left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \right\|$$

which is the Euclidean distance between the corresponding points in $\mathbb{R}^2$.

Sets:

$D(z_0; r) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$ "*open disk of radius* r" "*r-neighborhood of* z_0" $(r > 0)$

$D(z_0; r) \setminus \{z_0\} := \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$ "*deleted open disk*" $(r > 0)$

$A \subseteq \mathbb{C}$ is a "*neighborhood of* z_0" if and only if $\exists\, r > 0$ such that $D(z_0; r) \subseteq A$.

$A \subseteq \mathbb{C}$ is "*open*" iff $\forall\, z_0 \in A\ \exists\, r > 0$ s.t. $D(z_0; r) \subseteq A$.

For $F \subseteq \mathbb{C}$ the "*complement of F*" is defined as $\mathbb{C} \setminus F := \{z \in \mathbb{C}$ s.t. $z \notin F\}$.

$B \subseteq \mathbb{C}$ is "*closed*" iff its complement $\mathbb{C} \setminus B$ is open.

Theorem for open sets:
a) $\varnothing, \mathbb{C}$ are open;
b) The union of any collection of open sets is open;
c) The intersection of any finite collection of open sets is open.

Using DeMorgan's laws that complements of unions are intersections of complements; and that complements of intersections are unions of complements:

$$\mathbb{C} \setminus \left(\bigcup_{\gamma \in \Gamma} A_\gamma \right) = \bigcap_{\gamma \in \Gamma} \left(\mathbb{C} \setminus A_\gamma \right)$$

$$\mathbb{C} \setminus \left(\bigcap_{\gamma \in \Gamma} A_\gamma \right) = \bigcup_{\gamma \in \Gamma} \left(\mathbb{C} \setminus A_\gamma \right)$$

yields

Theorem for closed sets (that you've seen):
a) $\varnothing, \mathbb{C}$ are open;
b) The intersection of any collection of closed sets is closed;
c) The union of any finite collection of closed sets is closed.

Some further special properties of sets

$A \subseteq \mathbb{C}$ is *"bounded"* iff $\exists\ N \in \mathbb{R}$ s.t. $|z| \leq N \quad \forall\ z \in A$.

An *"open cover of A"* is a collection of open sets whose union contains A.

$K \subseteq \mathbb{C}$ is *"compact"* iff every cover of K by open sets has a *"finite subcover"*, i.e. a finite subcollection of the original collection already covers K.

A set $C \subseteq \mathbb{C}$ is *"not connected"* (or *"has a disconnection"*) iff $\exists\ U, V$ open subsets of $\mathbb{C}$ such that

$$C \subseteq U \cup V$$

$$C \cap U \neq \varnothing, \quad C \cap V \neq \varnothing$$

$$(C \cap U) \cap (C \cap V) = \varnothing$$

all hold.

A set $C \subseteq \mathbb{C}$ is *"connected"* iff it has no disconnection.

Sequences

$\epsilon - \delta$ definitions:

$\{z_k\} \to L$ iff
(note: limits are unique.)

$\{z_k\}$ is *"Cauchy"* iff

Theorem The sequence $\{z_k\}$ is Cauchy iff $\{z_k\}$ converges to some limit L.

Theorem If $\{z_k\} \to L$ and $\{w_k\} \to M$ and $a \in \mathbb{C}$ then

(i) $\{a\, z_k\} \to a\, L$

(ii) $\{z_k + w_k\} \to L + M$

(iii) $\{z_k\, w_k\} \to L\, M$

(iv) $\left\{\dfrac{z_k}{w_k}\right\} \to \dfrac{L}{M}$ provided $w_k \neq 0\ \forall\ k$ and $M \neq 0$.

sequences and sets

Def Let $B \subseteq \mathbb{C}$. $z \in \mathbb{C}$ is a *"limit point"* of B iff $\exists\ \{z_k\} \subseteq B$ s.t. $z_k \to z$.

Theorem $B \subseteq \mathbb{C}$ is closed iff B contains all of its limit points.

Def Let $B \subseteq \mathbb{C}$. The *"closure of B"* is the union of B with all of its limit points.

Theorem Let $B \subseteq \mathbb{C}$. Then the closure of B is closed, and can also be characterized as the intersection of all closed sets containing B.

Theorem The following are equivalent for $K \subseteq \mathbb{C}$.

(i) K is compact .

(ii) K is closed and bounded.

(iii) Every sequence $\{z_k\} \subseteq K$ has a convergent subsequence $\{z_{k_j}\}$ with $\{z_{k_j}\} \to z_0 \in K$.

proof: $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are reasonable to reconstruct in class. $(iii) \Rightarrow (i)$ is somewhat more intricate, so I'll include a proof on the next page which you can read or recall at your leisure, if you wish.

$(iii) \Rightarrow (i)$. We assume $K \subseteq \mathbb{C}$ has the property that every sequence $\{z_k\} \subseteq K$ has a convergent subsequence $\left\{z_{k_j}\right\}$ with $\left\{z_{k_j}\right\} \to z_0 \in K$. We wish to show that every open cover of K has a finite subcover. We will use the fact that the rational points $\{p + i\,q \mid p, q \in \mathbb{Q}\}$ are dense in $\mathbb{C}$, specifically the fact that $\mathbb{C}$ is separable:

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of K. For each $z \in U_\alpha$ pick $z_p, r_p \in \mathbb{Q}$ s.t $z \in \mathrm{D}(z_p, r_p) \subseteq U_\alpha$. Then the collection $\{\mathrm{D}(z_p, r_p)\}_{z \in U_\alpha, \alpha \in A}$ is a highly redundant countable cover of K. Re-label the non-redundant disks by the natural numbers,

$$\{\mathrm{D}(z_k, r_k)\}_{k \in \mathbb{N}}.$$

If we can find a finite subcover of K using these disks, then since each disk $\mathrm{D}(z_k, r_k)$ is in some U_{α_k}, the finite collection $\left\{U_{\alpha_k}\right\}$ will also be a finite subcover of K, and the theorem will be proven. We prove this fact by contradiction:

If no finite subcover exists, then for each $n \in \mathbb{N}$ pick

$$w_n \in K, w_n \notin \cup_{k=1}^{n} \mathrm{D}(z_k, r_k).$$

By the assumption (iii) a subsequence $\left\{w_{n_j}\right\} \to w_0 \in K$. But this $w_0 \in \mathrm{D}(z_k, r_k)$ for some fixed k since the collection of all these disks is a cover of K. Thus by convergence, there exists J s.t. $j \geq J \Rightarrow w_{n_j} \in \mathrm{D}(z_k, r_k)$. This violates how w_{n_j} was chosen, as soon as $n_j > k$. This contradiction proves that for some N,
$K \subseteq \cup_{k=1}^{N} \mathrm{D}(z_k, r_k) \subseteq \cup_{k=1}^{N} U_{\alpha_k}$.