Math 4200-001 Week 2 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes will represent an in-depth outline of what we plan to cover. This week we plan to cover sections 1.3-1.4.

Monday August 26 Finish discussing section 1.3, complex transformations.

Announcements

Warm-up exercise

On Friday we began discussing complex transformations f from $\mathbb{C} \to \mathbb{C}$. Using polar form we saw that the affine transformations

$$f(z) = a\,z + b$$

are compositions of (1) a rotation, followed by (2) a scaling, followed by (3) a translation: Writing

$$z = |\,z\,|\,\mathrm{e}^{i\,\theta}, \qquad \theta = arg(z)$$
$$a = |a\,|\,\mathrm{e}^{i\,\phi}, \qquad \phi = arg(a),$$

we wrote

$$f(z) = |a\,|\mathrm{e}^{i\,\phi}\,|z\,|\mathrm{e}^{i\,\theta} + b = |a\,|\,|\,z\,|\mathrm{e}^{i\cdot(\theta + \phi)} + b.$$
$$f = f_3 \circ f_2 \circ f_1$$
$$f_1(z) = \mathrm{e}^{i\,\theta} z \qquad \text{rotate}$$
$$f_2(w) = |a|\,w \qquad \text{scale}$$
$$f_3(q) = q + b \qquad \text{translate.}$$

When we start discussing complex differentiability in section 1.5 we will want to go back and forth between discussions for complex-valued functions $f: \mathbb{C} \to \mathbb{C}$ and the mathematically equivalent discussions for related real functions $F: \mathbb{R}^2 \to \mathbb{R}^2$ when we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. We will do this to take advantage of what you know about continuity and differentiability for real vector-valued functions of several variables, Math 3220.

The correspondence is that for each

$$f: \mathbb{C} \to \mathbb{C}$$
$$f(z) = f(x + i y) = \mathrm{Re}(f(x + i y)) + i\,\mathrm{Im}(f(x + iy))$$
$$:= u(x, y) + i\,v(x, y)$$

there is

$$F: \mathbb{R}^2 \to \mathbb{R}^2$$
$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

And for each such $F: \mathbb{R}^2 \to \mathbb{R}^2$ there is a corresponding $f: \mathbb{C} \to \mathbb{C}$ given by $f(x + i y) := u(x, y) + i\,v(x, y)$.

Example 1 apply this correspondence to the example on the previous page, with

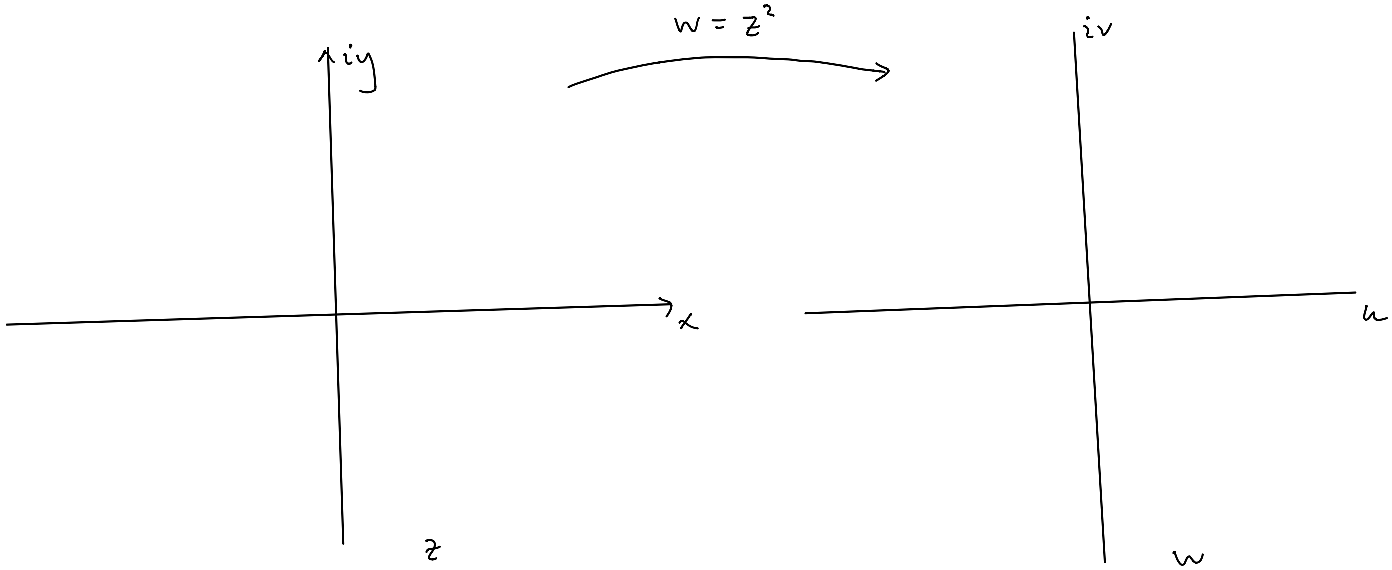
$$f(z) = a z + b$$
$$z = x + i y$$
$$a = a_1 + i a_2$$
$$b = b_1 + i b_2.$$

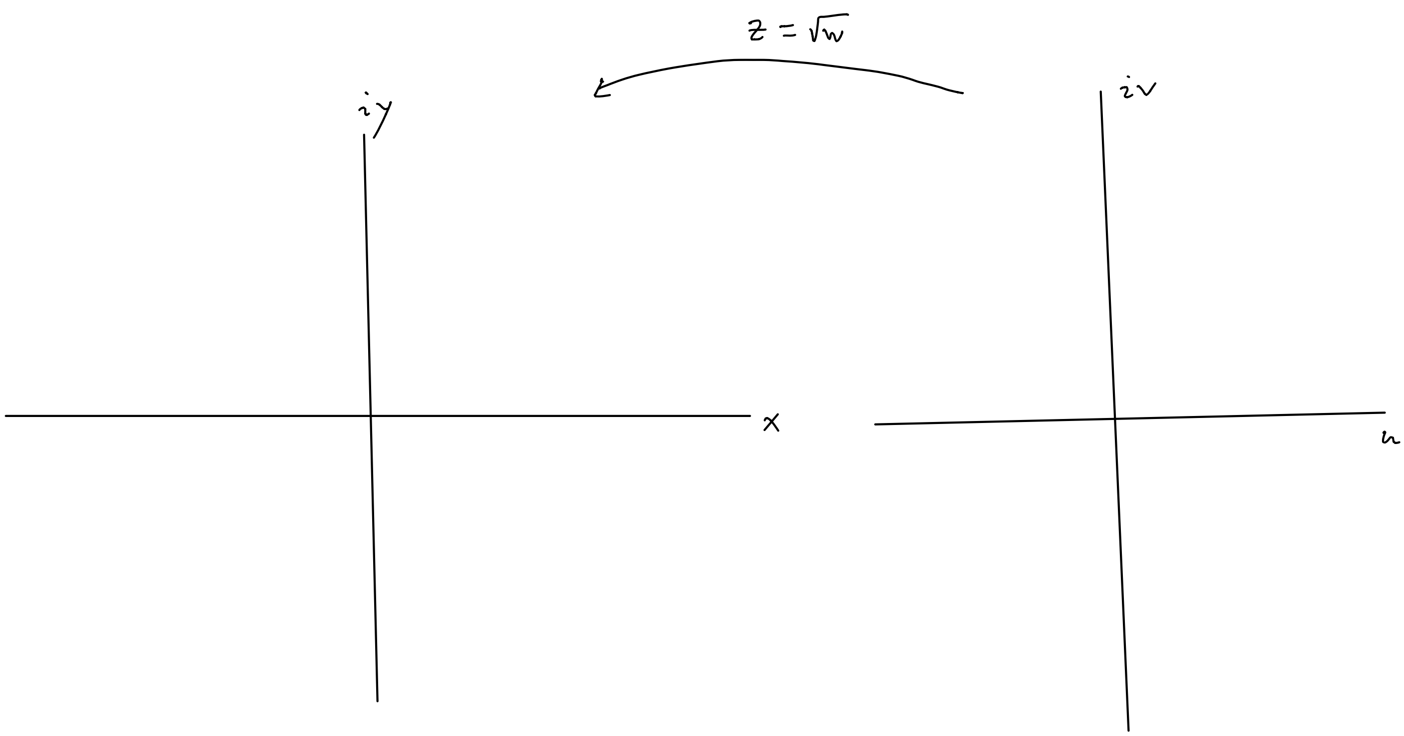
Find the corresponding $F(x, y)$ and recall your linear transformations of the plane from linear algebra, in order to recover the same geometric description as on the previous page, but this time in terms of the transformation $F: \mathbb{R}^2 \to \mathbb{R}^2$.

Example 2

$$f(z) = z^2$$
$$f\left(r\,e^{i\,\theta}\right) = r^2\,e^{i\,2\,\theta}$$

Discuss and sketch how f transforms z - plane into a (mostly) twice-covered w - plane. For $w = z^2$ discuss possible continuous inverse functions $z = \sqrt{w}$, and corresponding domains.

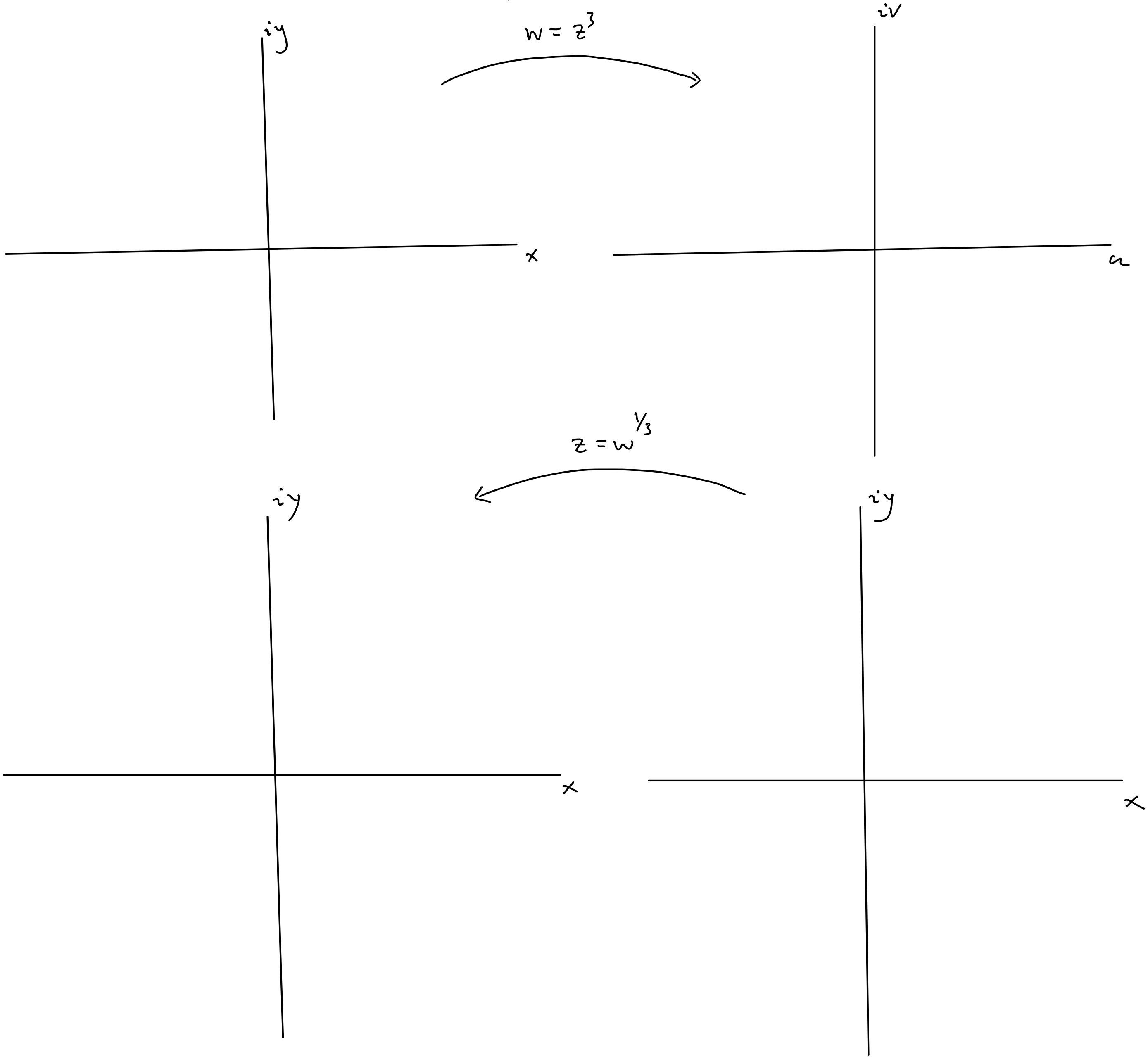




Example 3

$$f(z) = z^3$$
$$f\left(r\,e^{i\,\theta}\right) = r^3 e^{i\,\theta}.$$

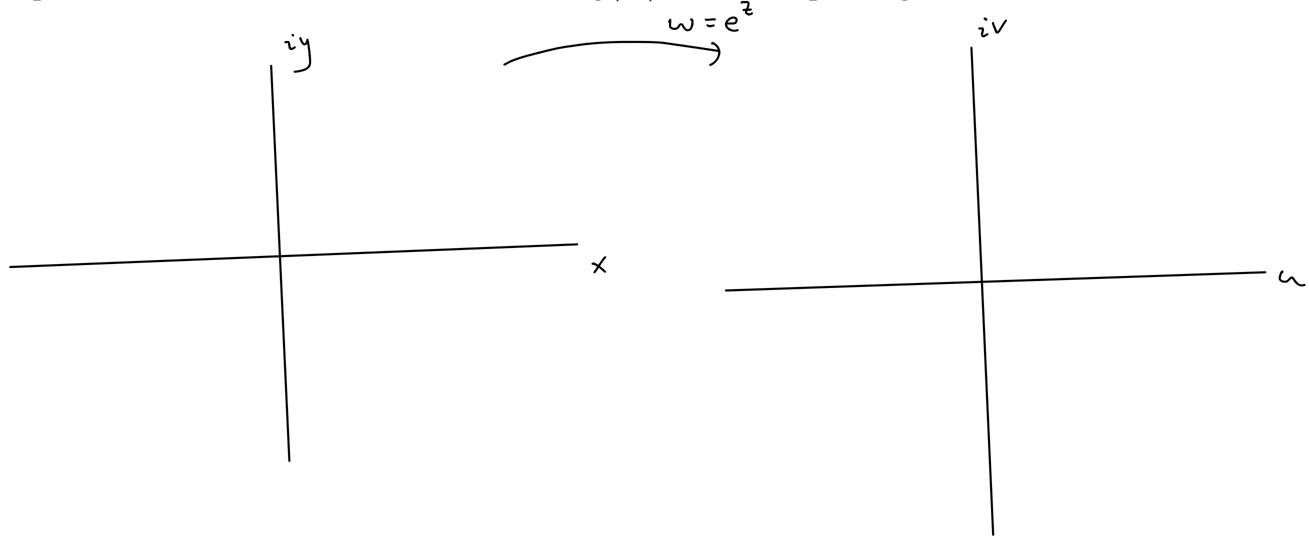
Discuss and sketch how f transforms the z - plane into a (mostly) triple-covered w - plane. For $w = z^3$ discuss possible continuous inverse functions $z = \sqrt[3]{w}$, and corresponding domains.

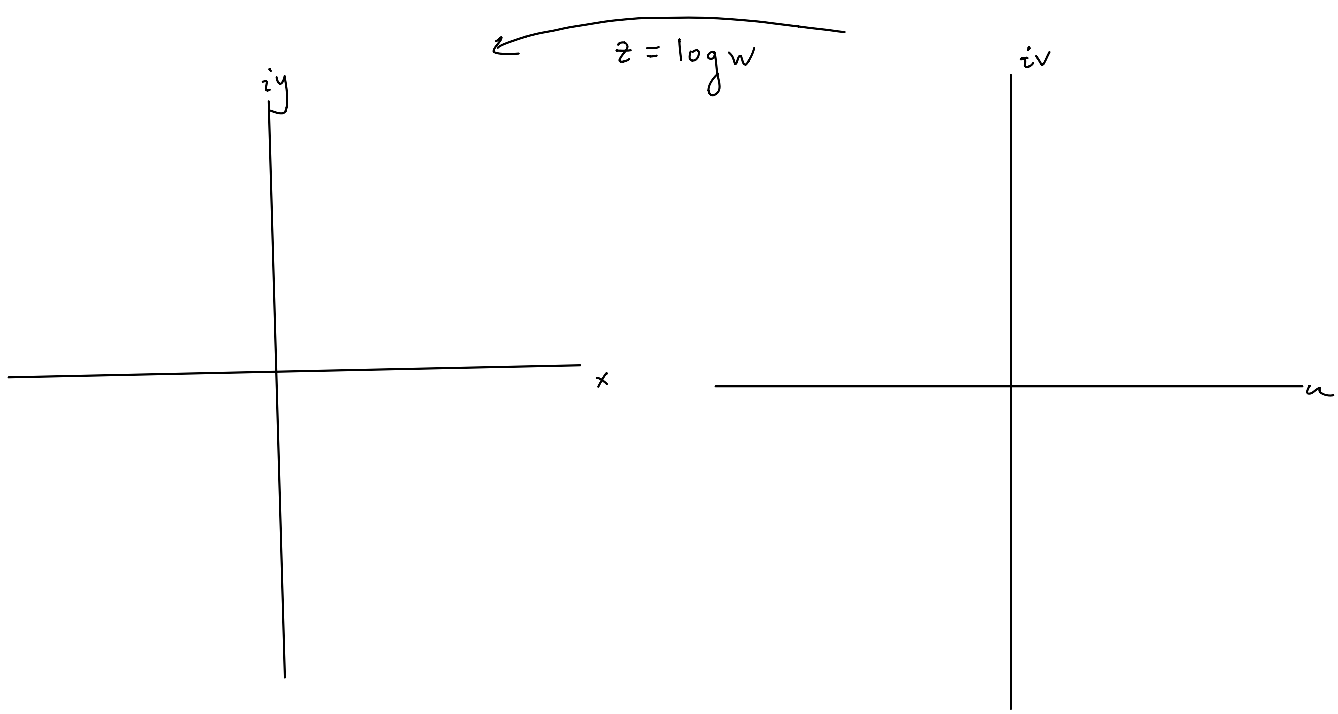


Example 4 For $z = x + i\,y, \quad x, y \in \mathbb{R}$

$$f(z) = e^{z} = e^{x + i\,y} := e^{x}\, e^{i\,y}$$

Discuss and sketch how f transforms the z-plane into an infinitely covered w-plane. For $w = e^{z}$ discuss possible continuous inverse functions $z = \log(w)$, and corresponding domains.





Example 5 "Trig functions".
If x is real,

$$\text{eqtn 1} \qquad e^{ix} = \cos(x) + i\sin(x)$$
$$\text{eqtn 2} \qquad e^{-ix} = \cos(x) - i\sin(x).$$

$\dfrac{eqtn\ 1 + eqtn2}{2} \Rightarrow$

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$\dfrac{eqtn\ 1 - eqtn2}{2\,i} \Rightarrow$

$$\sin(x) = \frac{1}{2\,i}\left(e^{ix} - e^{-ix}\right).$$

Also recall the hyperbolic trig functions

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{x}\right)$$
$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$

So we define, for $z \in \mathbb{C}$,

$$\cos(z) := \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \cosh(z) := \frac{1}{2}\left(e^{z} + e^{-z}\right) = \cos(iz)$$
$$\sin(z) = \frac{1}{2\,i}\left(e^{iz} - e^{-iz}\right) \qquad \sinh(z) := \frac{1}{2}\left(e^{z} - e^{-z}\right) = -i\sin(iz).$$

"Trig" identities hold, via properties of complex exponential multiplication. (but note that $\sin(z)$, $\cos(z)$ are no longer bounded functions....it's challenging to figure out their transformation pictures like we did for the earlier examples).

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

$$\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

trigh ...

$$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) + \sin^2(iz) = 1 \quad .....$$