Start here Friday

Solving polynomial equations in $\mathbb{C}$.

$e^{i(\frac{\theta}{2})} \underbrace{e^{i\pi}}_{-1}$

1) Every non-zero complex number z_0 has two square roots, i.e solutions z to

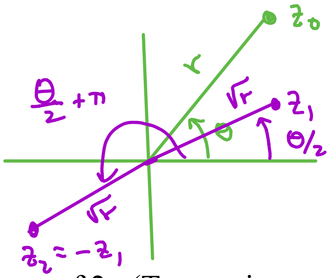
$$z^2 = z_0$$

and they are opposites.

so $z = \sqrt{r}\, e^{i\frac{\theta}{2}}, \sqrt{r}\, e^{i(\frac{\theta}{2}+\pi)}$

all other θ's yield one of these two.

proof 1: Use the polar form, writing $z_0 = r\, e^{i\theta}$, $z = \rho\, e^{i\phi}$, with $r, \rho > 0$.



$z^2 = (\rho e^{i\phi})^2 = \rho^2 e^{i(2\phi)} = r e^{i\theta}$

modulus: $\rho^2 = r \Rightarrow \rho = \sqrt{r}$

arg: $2\phi = \theta + 2\pi k, \ k \in \mathbb{Z}$

$\phi = \frac{\theta}{2} + \pi k, \ k \in \mathbb{Z}$

proof 2: (To convince you how great polar form is) Use rectangular coordinates: Express z_0, z in terms of their real and imaginary parts,

$$z_0 = x_0 + i\, y_0$$
$$z = x + i\, y$$
$$(x + i\, y)^2 = x_0 + i\, y_0$$
$$\begin{cases} x^2 - y^2 = x_0 \\ 2\, x\, y = y_0 \end{cases}$$

Case 1: If $y_0 \neq 0$ then $x, y \neq 0$. Solve for y from the second equation and subsitute into the first:

$$x^2 - \left(\frac{y_0}{2\, x}\right)^2 = x_0$$
$$4\, x^4 - 4\, x_0 x^2 - y_0^2 = 0.$$

Use the quadratic formula for real coefficients for x^2 and throw out the negative value:

$$x^2 = \frac{4\, x_0 + \sqrt{16\, x_0^2 + 16\, y_0^2}}{8} = \frac{x_0 + \sqrt{x_0^2 + y_0^2}}{2}.$$

There are two opposite real values of x which solve this equation, with corresponding opposite values of $y = \frac{y_0}{2\, x}$.

Case 2: If $y_0 = 0$ it meant that $z_0 = x_0$ was real, and you already know how to find the two square roots. If $x_0 > 0$ they will be real square roots, and if $x_0 < 0$ they will be imaginary.

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2) Every quadratic equation has 2 ~~roots~~ solutions (counting multiplicity).

$a z^2 + b z + c = 0, \quad a \neq 0$

using field axioms for $\mathbb{C}$, complete square, take square roots to find

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

← 2 possible square roots from step 1)

(unless $b^2 - 4ac = 0$, in which case $z = -\frac{b}{2a}$ double root)

3) The general degree n polynomial equation

$$p(z) := z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 = 0.$$

$$a_{n-1}, \dots a_1, a_0 \in \mathbb{C}$$

You've been told forever that every degree n polynomial equation has n complex roots, counting multiplicty. This fact is known as "The Fundamental Theorem of Algebra." You'll learn a beautiful "elementary" proof of the fundamental theorem of algebra in this class. It's a proof by contradiction though, and except for very special polynomials there are no explicit formulas for exact solutions.....

I told you ~~yesterday~~ Monday that there is a cubic formula for cubic equations. There is also a formula for the roots of 4^{th} order polynomials. The Abel-Ruffini Theorem asserts however, that there ~~is~~ are no general formulas for the roots of degree 5 and higher polynomial equations, such that these formulas use only the algebraic operations of addition, multiplication, and taking radicals (square roots, cube roots, etc.). One of the founders of number theory, Évariste Galois, developed "Galois Theory", which explains exactly which higher degree polynomial equations can be solved using these operations. These are topics in advanced algebra courses.

4) The special polynomial equation $z^n = 1$.

Its solutions are called "the n^{th} roots of unity", and there are n of them.

HW
$(z+w)^n = z^n + \binom{n}{1} z^{n-1} w + \ldots$

Since complex multiplication in polar form reads

$$z\,w = r\,e^{i\,\theta}\,\rho\,e^{i\,\phi} = r\,\rho\,e^{i\cdot(\theta+\phi)},$$

(where $r = |z|$, $\theta = arg(z)$, $\rho = |w|$, $\phi = arg(w)$), it's easy to check via induction, that

$$z^n = r^n\,e^{i\,n\,\theta}.$$

This formula for powers is known as "DeMoivre's formula".

by induction:

- for $n=1$ formula reads $z = re^{i\theta}$
  ($n=2$: $z^2 = re^{i\theta} re^{i\theta} = r^2 e^{i(2\theta)}$ by rule for mult. zw)
- Assume true for $n=k$. Use assumption to show true for $n=k+1$.
  $z^{k+1} = z^k z = (re^{i\theta})^k re^{i\theta} = (r^k e^{ik\theta}) re^{i\theta}$ (use ind. hyp)
  $= r^{k+1} e^{i(k+1)\theta}$ using zw formula. ∎

So, to solve $z^n = 1$, express $z = |z|\,e^{i\,\theta}$ and solve

$$z^n = |z|^n\,e^{i\,n\,\theta} = 1. = 1\,e^{i0}$$

moduli: $|z|^n = 1 \Rightarrow |z| = 1$

$n\theta = 0 + 2\pi k \quad k \in \mathbb{Z}$

$\theta = 2\pi\left(\frac{k}{n}\right)$

$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots 2\pi\left(\frac{n-1}{n}\right), 2\pi, \ldots$

up to a mult of 2π, each θ is one of these.

$z = e^{i0}, e^{i\frac{2\pi}{n}}, \ldots e^{i\left(\frac{2\pi}{n}\right)(n-1)}$

yields n equally distributed point on unit circle, as in warm-up ex.

At the end of section 1.2, our text lists a number of identities and estimates that we'll use going forward.... we'll be doing analysis soon and will be using the triangle inequality, for example. We've already seen some of these estimates, and can discuss the ones that seem new.

Write $z = x + i\,y$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$. So $\overline{z} = x - i\,y$.

1) $z + w = \overline{z} + \overline{w}$

2) $\overline{zw} = \overline{z}\,\overline{w}$ $\qquad \left(\frac{z}{w}\right) = \frac{\overline{z}}{\overline{w}}$

3) $|z|^2 = z\,\overline{z}$

4) $z = \overline{z}$ if and only if z is real. $\qquad z = -\overline{z}$ if and only if z is imaginary.

5) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$

$\mathrm{Im}(z) = \frac{1}{2\,i}(z - \overline{z})$

6) $\overline{\overline{z}} = z$

7) $|z\,w| = |z|\,|w|$ $\qquad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

8) $-|z| \le \mathrm{Re}(z) \le |z|$ $\qquad$ i.e. $|\mathrm{Re}(z)| \le |z|$

$-|z| \le \mathrm{Im}(z) \le |z|$ $\qquad$ i.e. $|\mathrm{Im}(z)| \le |z|$

9) $|\overline{z}| = |z|$

10) $|z + w| \le |z| + |w|$ $\qquad$ triangle inequality

11) $|z - w| \ge |z| - |w|$ $\qquad$ reverse triangle inequality

12) $\left|\sum_{j=1}^{n} z_j\,w_j\right|^2 \le \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right)$ $\qquad$ complex ~~Cauch~~ Cauchy-Schwarz.

Math 4200
Friday August 23
1.2-1.3 Algebra and geometry of complex arithmetic from Wednesday's notes; introduction to complex plane transformations section 1.3, in today's notes. We'll pick up in Wednesday's notes where we left off, and continue into today's.

Announcements: After today I'll just bring current notes & assume you have the previous day's if necessary.

- Except – each Monday I'll start fresh.
- HW due Wednesday!
  http://www.math.utah.edu/~korevaar/4200fall19
  after today you have the necessary knowledge for almost all the problems
- CANVAS soon

Warm-up exercise: Find all $z \in \mathbb{C}$ for which $z^5 = 1$

Hint: write $z = |z| e^{i\theta}$

Sketch these five "fifth roots of unity" in $\mathbb{C}$

$$z^5 = \left(|z| e^{i\theta}\right)^5 = |z|^5 e^{i5\theta} = 1 e^{i0}$$

$\Longrightarrow |z|^5 = 1$ (comparing moduli)

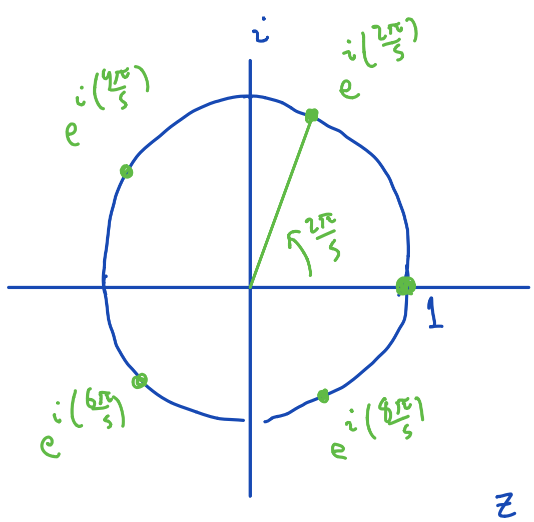
$5\theta = 0 + 2\pi k \quad k \in \mathbb{Z}$

$\Longrightarrow |z| = 1$

$\theta = \frac{2\pi k}{5} \quad k \in \mathbb{Z}$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \cancel{\frac{10\pi}{5}}$ (2π)

[all possible θ's are congruent to these modulo 2π]



$z = 1e^{i0}, 1e^{i\frac{2\pi}{5}}, 1e^{i\frac{4\pi}{5}}, e^{i\frac{6\pi}{5}}, e^{i\frac{8\pi}{5}}$

(all other acceptable θ's yield one of these 5)

Section 1.3, "basic" complex functions.

<u>Example 1</u> $f:\mathbb{C}\to\mathbb{C},\ f(z) = a\,z + b \quad (a, b \in \mathbb{C})$.

Using polar form makes this transformation easy to understand geometrically. Write

$$\begin{cases} z = |z|\,e^{i\theta}, & \theta = arg(z) \\ a = |a|\,e^{i\phi}, & \phi = arg(a). \end{cases}$$

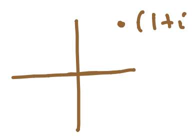
and compute $f(z)$ in order to interpret it as the composition of a rotation, a scaling, and then a translation.

$$f(z) = az + b$$
$$= |a|e^{i\phi}\,|z|e^{i\theta} + b$$
$$= |a||z|e^{i(\theta+\phi)} + b$$

① rotated z by ϕ

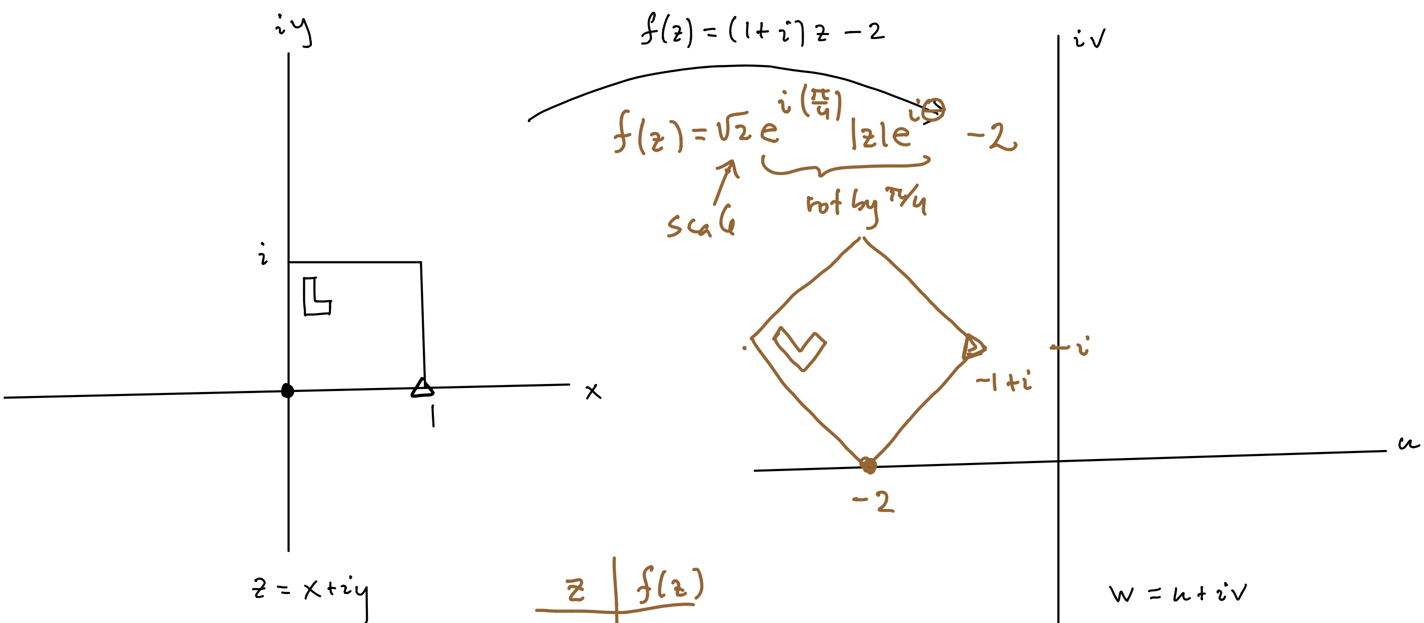
② scaled by $|a|$

③ translated by b.



$\sqrt{2}e^{i\pi/4}$

Illustrate the general situation above for the transformation $f(z) = (1+i)\,z - 2$.



z	$f(z)$
0	-2
1	$1+i-2 = -1+i$