Solving polynomial equations in $\mathbb{C}$.

1) Every non-zero complex number z_0 has two square roots, i.e solutions z to

$$z^2 = z_0$$

and they are opposites.

<u>proof 1</u>: Use the polar form, writing $z_0 = r\,e^{i\,\theta}$, $z = \rho\,e^{i\,\phi}$, with $r, \rho > 0$.

<u>proof 2</u>: (To convince you how great polar form is) Use rectangular coordinates: Express z_0, z in terms of their real and imaginary parts,

$$z_0 = x_0 + i\,y_0$$
$$z = x + i\,y$$
$$(x + i\,y)^2 = x_0 + i\,y_0$$
$$\begin{cases} x^2 - y^2 = x_0 \\ 2\,x\,y = y_0 \end{cases}$$

Case 1: If $y_0 \neq 0$ then $x, y \neq 0$. Solve for y from the second equation and subsitute into the first:

$$x^2 - \left(\frac{y_0}{2\,x}\right)^2 = x_0$$
$$4\,x^4 - 4\,x_0 x^2 - y_0^2 = 0.$$

Use the quadratic formula for real coefficients for x^2 and throw out the negative value:

$$x^2 = \frac{4\,x_0 + \sqrt{16\,x_0^2 + 16\,y_0^2}}{8} = \frac{x_0 + \sqrt{x_0^2 + y_0^2}}{2}.$$

There are two opposite real values of x which solve this equation, with corresponding opposite values of $y = \dfrac{y_0}{2\,x}$.

Case 2: If $y_0 = 0$ it meant that $z_0 = x_0$ was real, and you already know how to find the two square roots. If $x_0 > 0$ they will be real square roots, and if $x_0 < 0$ they will be imaginary.

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2) Every quadratic equation has 2 roots (counting multiplicity).

3) The general degree n polynomial equation

$$p(z) := z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0.$$

$$a_{n-1}, \ldots a_1, a_0 \in \mathbb{C}$$

You've been told forever that every degree n polynomial equation has n complex roots, counting multiplicity. This fact is known as "The Fundamental Theorem of Algebra." You'll learn a beautiful "elementary" proof of the fundamental theorem of algebra in this class. It's a proof by contradiction though, and except for very special polynomials there are no explicit formulas for exact solutions.....

I told you ~~yesterday~~ Monday that there is a cubic formula for cubic equations. There is also a formula for the roots of 4^{th} order polynomials. The Abel-Ruffini Theorem asserts however, that there ~~is~~ are no general formulas for the roots of degree 5 and higher polynomial equations, such that these formulas use only the algebraic operations of addition, multiplication, and taking radicals (square roots, cube roots, etc.). One of the founders of number theory, Évariste Galois, developed "Galois Theory", which explains exactly which higher degree polynomial equations can be solved using these operations. These are topics in advanced algebra courses.

4) The special polynomial equation $z^n = 1$.
Its solutions are called "the n^{th} roots of unity", and there are n of them.

Since complex multiplication in polar form reads

$$z\,w = r\,e^{i\,\theta}\,\rho\,e^{i\,\phi} = r\,\rho\,e^{i\cdot(\theta+\phi)},$$

(where $r = |z|$, $\theta = arg(z)$, $\rho = |w|$, $\phi = arg(w)$), it's easy to check via induction, that

$$z^n = r^n\,e^{i\,n\,\theta}.$$

This formula for powers is known as "DeMoivre's formula".

So, to solve $z^n = 1$, express $z = |z|\,e^{i\,\theta}$ and solve

$$|z|^n\,e^{i\,n\,\theta} = 1.$$

At the end of section 1.2, our text lists a number of identities and estimates that we'll use going forward.... we'll be doing analysis soon and will be using the triangle inequality, for example. We've already seen some of these estimates, and can discuss the ones that seem new.

Write $z = x + i\,y$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$. So $\overline{z} = x - i\,y$.

1) $z + w = \overline{z} + \overline{w}$

2) $z\overline{w} = \overline{z}\;\overline{w}$ $\qquad \left(\frac{z}{w}\right) = \frac{\overline{z}}{\overline{w}}$

3) $|z|^2 = z\,\overline{z}$

4) $z = \overline{z}$ if and only if z is real. $\qquad z = -\overline{z}$ if and only if z is imaginary.

5) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$

$\mathrm{Im}(z) = \frac{1}{2\,i}(z - \overline{z})$

6) $\overline{\overline{z}} = z$

7) $|z\,w| = |z|\,|w|$ $\qquad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

8) $-|z| \leq \mathrm{Re}(z) \leq |z|$ $\qquad$ i.e. $|\mathrm{Re}(z)| \leq |z|$

$-|z| \leq \mathrm{Im}(z) \leq |z|$ $\qquad$ i.e. $|\mathrm{Im}(z)| \leq |z|$

9) $|\overline{z}| = |z|$

10) $|z + w| \leq |z| + |w|$ triangle inequality

11) $|z - w| \geq |z| + |w|$ reverse triangle inequality

12) $\left|\sum_{j=1}^{n} z_j\, w_j\right|^2 \leq \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right)$ complex Cauch-Schwarz.

Math 4200
<u>Friday August 23</u>
1.2-1.3 Algebra and geometry of complex arithmetic from Wednesday's notes; introduction to complex plane transformations section 1.3, in today's notes. We'll pick up in Wednesday's notes where we left off, and continue into today's.

Announcements:

Warm-up exercise:

Section 1.3, "basic" complex functions.

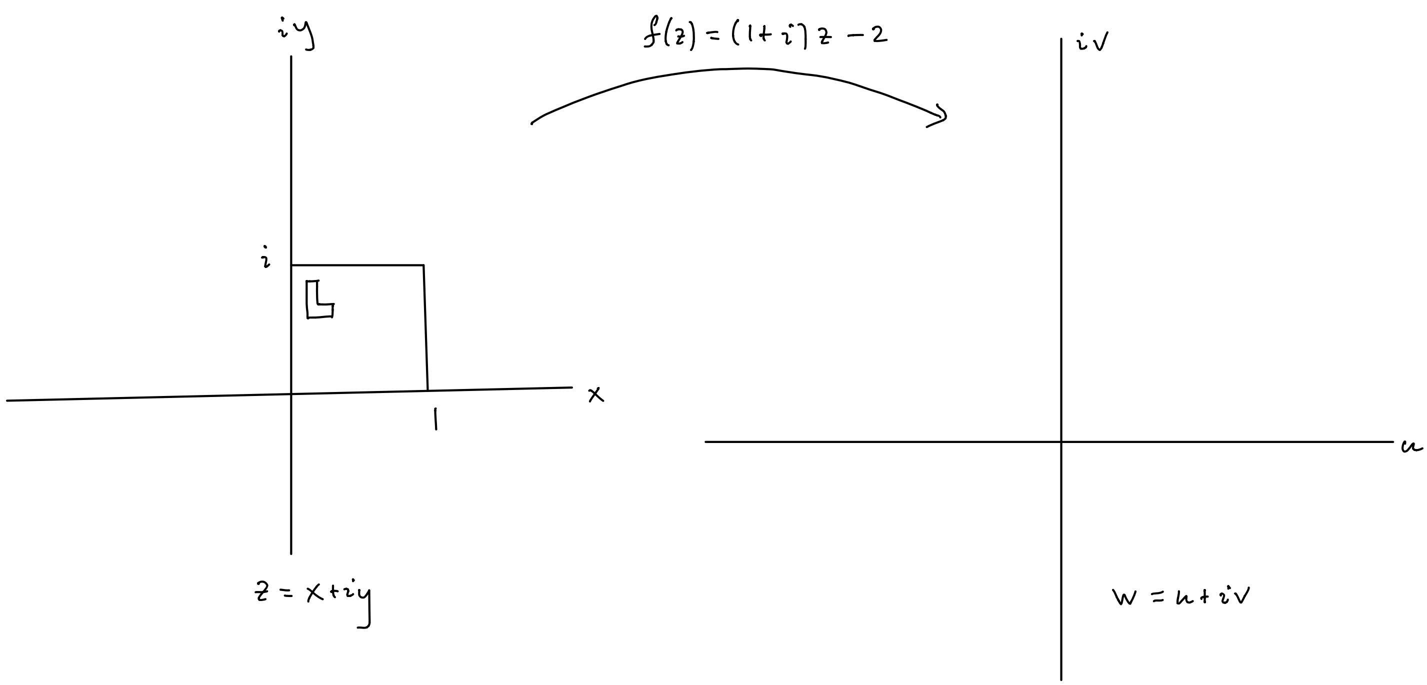
Example 1 $f: \mathbb{C} \to \mathbb{C}$, $f(z) = a\,z + b \quad (a, b \in \mathbb{C})$.

Using polar form makes this transformation easy to understand geometrically. Write

$$z = |\,z\,|\,\mathrm{e}^{i\,\theta}, \qquad \theta = arg(z)$$
$$a = |a\,|\,\mathrm{e}^{i\,\phi}, \qquad \phi = arg(a).$$

and compute $f(z)$ in order to interpret it as the composition of a rotation, a scaling, and then a translation.

Illustrate the general situation above for the transformation $f(z) = (1 + i)\,z - 2$.



When we start discussing complex differentiability in section 1.5 we will want to go back and forth between discussions for complex-valued functions $f: \mathbb{C} \rightarrow \mathbb{C}$ and the mathematically equivalent discussions for related real functions $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, so that we can take advantage of what you know about continuity and differentiability for real vector-valued functions, Math 3220.

The correspondence is that for each

$$f: \mathbb{C} \rightarrow \mathbb{C}$$
$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$$

with $u(x, y) = \mathrm{Re}(f(z))$, $v(x, y) = \mathrm{Im}(f(z))$, there is

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

And for each $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ there is a corresponding $f: \mathbb{C} \rightarrow \mathbb{C}$.

If we apply this correspondence to the example on the previous page, with $f(z) = a\,z + b$ and

$$z = x + i\,y$$
$$a = a_1 + i\,a_2$$
$$b = b_1 + i\,b_2,$$

then

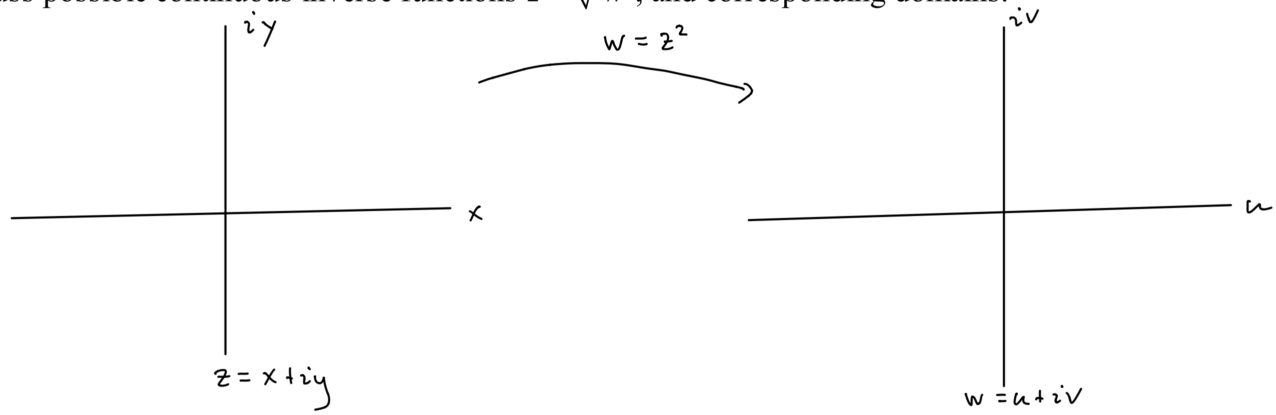
$$f(x + i\,y) = (a_1 + i\,a_2)(x + i\,y) + (b_1 + i\,b_2) = (a_1\,x - a_2\,y + b_1) + i(a_2\,x + a_1 y + b_2)$$
$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Re}(f) \\ \mathrm{Im}(f) \end{bmatrix} = \begin{bmatrix} a_1\,x - a_2\,y + b_1 \\ a_2\,x + a_1 y + b_2 \end{bmatrix}$$
$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

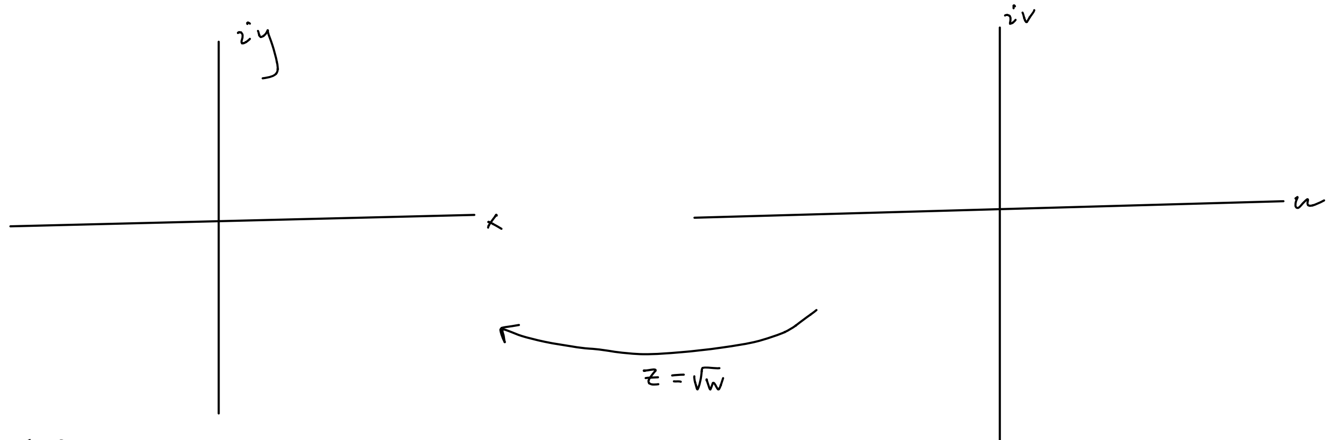
Recall your geometric matrix transformations of the plane from your linear algebra class to describe F geometrically, just as we described f on the previous page.

Example 2

$$f(z) = z^2$$
$$f\left(r\, e^{i\,\theta}\right) = r^2\, e^{i\,2\,\theta}$$

Discuss and sketch how f transforms z - plane into a (mostly) twice-covered w - plane. For $w = z^2$ discuss possible continuous inverse functions $z = \sqrt{w}$, and corresponding domains.

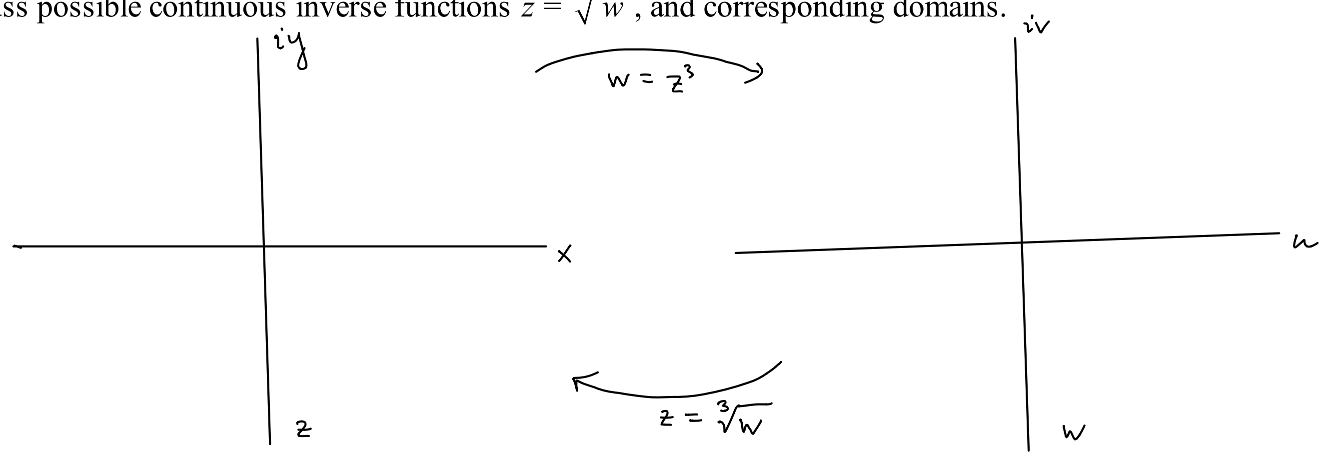




Example 3

$$f(z) = z^3$$
$$f\left(r\, e^{i\,\theta}\right) = r^3 e^{i\,\theta}.$$

Discuss and sketch how f transforms the z - plane into a (mostly) triple-covered w - plane. For $w = z^3$ discuss possible continuous inverse functions $z = \sqrt[3]{w}$, and corresponding domains.



<u>Example 4</u> For $z = x + i\,y$, $x, y \in \mathbb{R}$

$$f(z) = e^z = e^{x + i\,y} := e^x\, e^{i\,y}$$

Discuss and sketch how f transforms the z - plane into an infinitely covered w - plane. For $w = e^z$ discuss possible continuous inverse functions $z = \log(w)$, and corresponding domains.