Important operations for complex numbers:

Let $z = x + i\,y$ with $x, y \in \mathbb{R}$. Then

$Re(z) := x$ "Real part of z"

$Re(2+3i) = 2$

$Im(2+3i) = 3$

$Im(z) := y$ "Imaginary part of z" (even though it's a real number)

$\bar{z} := x - i\,y$ "conjugate of z" or "z bar".

$\overline{2+3i} = 2-3i$

$|z| := \sqrt{x^2 + y^2}$ "modulus of z" or "magnitude of z"

text says "absolute value"

$|2+3i| = \sqrt{13}$

corresponds to vector magnitudes

Check:

$\overline{zw} = \bar{z}\,\bar{w}$.

$\overline{(x+iy)(u+iv)} = \overline{(xu-yv)+i(xv+yu)} = (xu-yv)-i(xv+yu)$

$\bar{z}\bar{w} = (x-iy)(u-iv) = xu - yv + i(-xv-yv)$

$(-iy)(-iv)$

$-1yv$

✓

$x^2+y^2 = (x+iy)(x-iy)$

$|z|^2 = z\,\bar{z}$ so $|z| = \sqrt{z\,\bar{z}}$.

- $|zw|^2 = |z|^2|w|^2$: $|zw|^2 = zw(\overline{zw})$
$= zw\bar{z}\bar{w}$
$= z\bar{z}w\bar{w} = |z|^2|w|^2$

$\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}$: $z\left(\frac{\bar{z}}{|z|^2}\right) = \frac{z\bar{z}}{|z|^2} = 1$ ✓

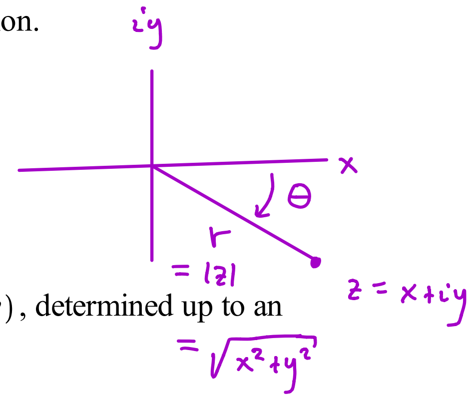
in practice: $\frac{1}{z}\,\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2}$

$\frac{1}{2+3i} = \frac{1}{2+3i}\cdot\frac{2-3i}{2-3i} = \frac{2-3i}{13}$

$\left(= \frac{2}{13} + i\left(\frac{-3}{13}\right)\right)$

Compare to our earlier example, where $z = 2 + 3\,i$.

Polar form of complex numbers and the geometric meaning of complex multiplication.



Recall polar coordinates in $\mathbb{R}^2$: Every non-zero vector in $\mathbb{R}^2$ can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle from the positive $x-$ axis to the point (x, y), determined up to an integer multiple of 2π.

Translating this to complex notation and using $arg(z)$ ("argument of z") for θ:

$$z = x + i\,y = r\cos(\theta) + i\,r\sin(\theta)$$

$$|z| = \sqrt{x^2 + y^2} = r$$

$$arg(z) := \theta.$$

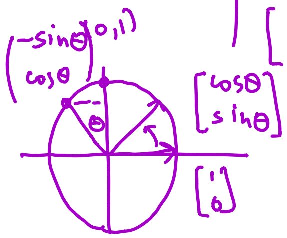
<u>Theorem</u>: Let $z = r\cdot(\cos(\theta) + i\sin(\theta))$ and $w = \rho\cdot(\cos(\phi) + i\sin(\phi))$ be complex numbers written in polar form. Then

$$z\,w = r\,\rho\,(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

In other words, when you multiply two complex numbers their moduli multiply and their arguments add.

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

$$\begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}}_{[Rot_\theta]} = [Rot_{\phi+\theta}] = \begin{bmatrix}\cos(\theta+\phi) & -- \\ \sin(\theta+\phi) & --\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi & \bullet \\ \cos\theta\sin\phi + \sin\theta\cos\phi & \bullet\end{bmatrix}$$



$$zw = r(\cos\theta + i\sin\theta)\,\rho(\cos\phi + i\sin\phi)$$
$$= r\rho\left[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\right]$$
$$= r\rho(\cos(\theta+\phi) + i\sin(\theta+\phi))$$

$$e^{i\theta} := \cos\theta + i\sin\theta$$

$$\left(e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots \right. \qquad e^{i\theta} := \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots}_{\cos\theta} + \underbrace{i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \cdots}_{i\sin\theta}$$

Note: If you recall Euler's formula from Math 2280 then the multiplication formula reads as follows: For

$$z = r\cdot(\cos(\theta) + i\sin(\theta)) = r\,e^{i\,\theta}$$

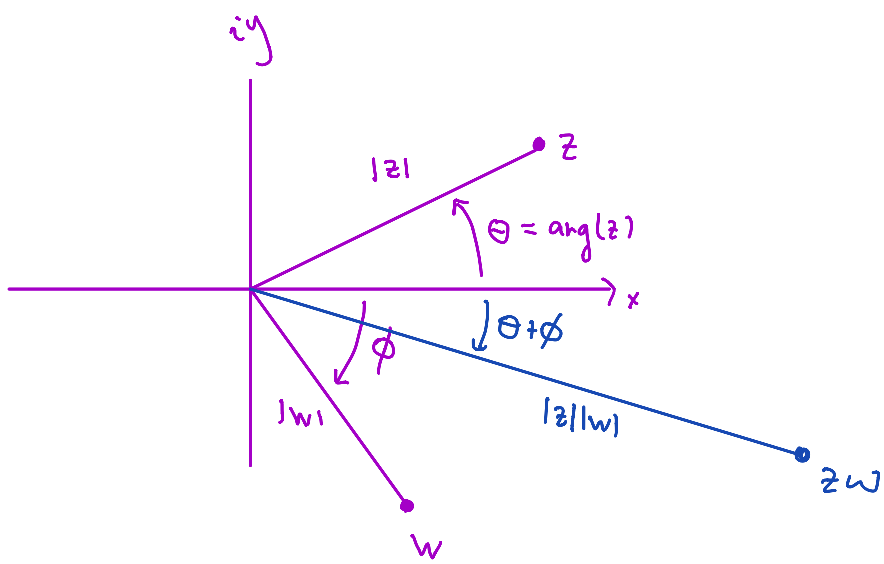
$$w = \rho\cdot(\cos(\phi + i\sin(\phi)) = \rho\,e^{i\,\phi},$$

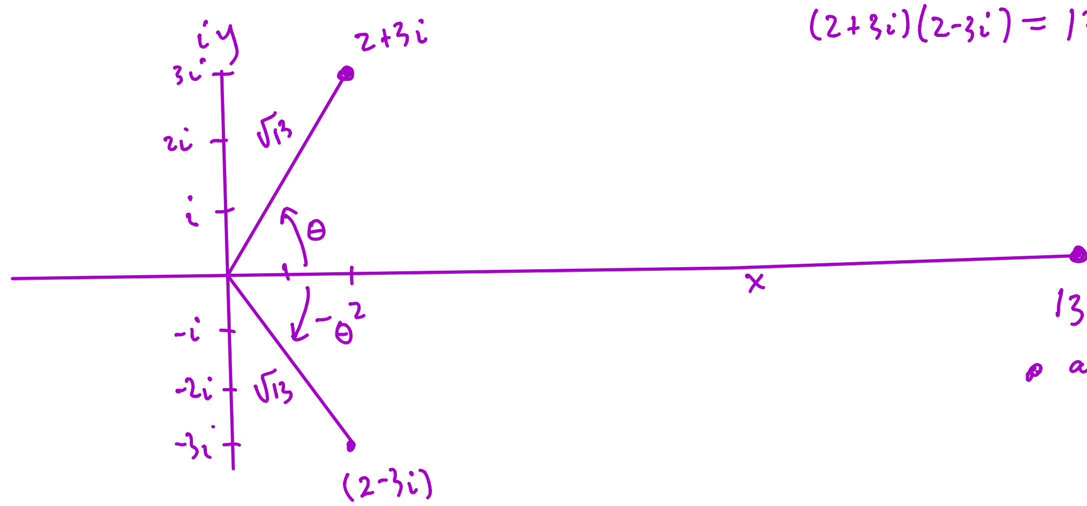
the product

$$z\,w = r\,\rho\,e^{i\cdot(\theta+\phi)}.$$

$$zw := \underset{|z|}{r}e^{i\theta}\,\underset{|w|}{\rho}e^{i\phi} \overset{?}{=} r\rho\, e^{i(\theta+\phi)}$$

checked!





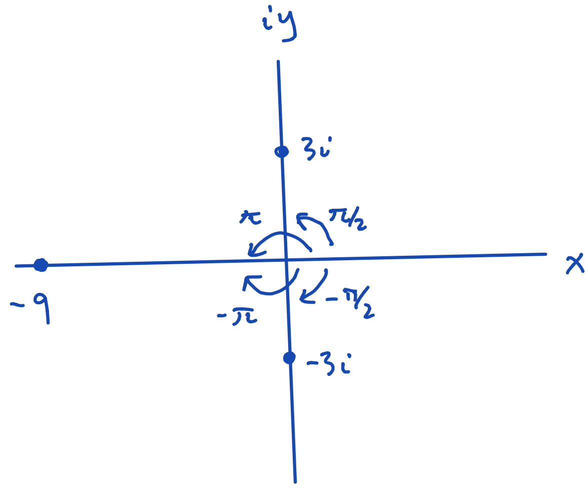
$(2+3i)(2-3i) = 13$

$13 = \sqrt{13}\sqrt{13}$

$\arg = \theta - \theta = 0$

Use the polar form of complex numbers for the following, and compare to the rectangular coordinates computation.

- Solve $z^2 = -9$ for z. Sketch.
- Find all solutions to $z^3 = 27$. Sketch.



$z^2 = -9$

rect $\quad z = \pm 3i$

polar form: $\quad z^2 = -9$

modulus, arg

let $z = re^{i\theta}$

$z^2 = r^2 e^{i2\theta} = 9 e^{i\pi}$

$| \, |: \quad r^2 = 9 \Rightarrow r = 3 \quad (r \geq 0)$

arg: $\quad 2\theta = \pi + 2\pi k \qquad k \in \mathbb{Z}$.

$\theta = \frac{\pi}{2} + \pi k \qquad k \in \mathbb{Z}$

$\theta = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3}{2}\pi, -\frac{3}{2}\pi \; \ldots$

e.g. $\theta = \frac{\pi}{2}, -\frac{\pi}{2}$ $\qquad$ rest differ by $2\pi k$

$z = 3e^{i\pi/2}$ or $3e^{-i\pi/2}$

$= 3i, -3i$

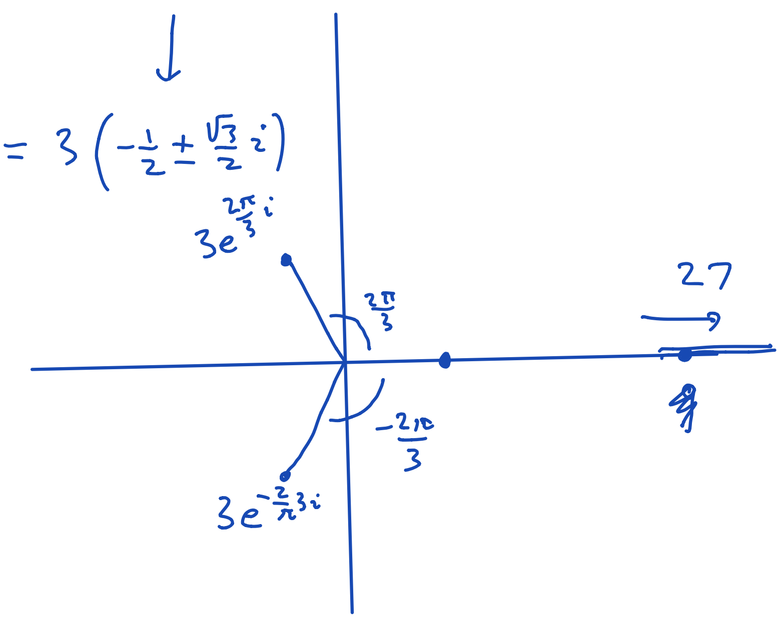
$z^3 = 27$

rect $\quad z^3 - 27 = 0$

$(z-3)(z^2 + 3z + 9) \qquad \sqrt{-27}$

$z = 3 \qquad z = \dfrac{-3 \pm \sqrt{9 - 36}}{2} = \dfrac{-3}{2} \pm \dfrac{3\sqrt{3}}{2} i \,! = 3\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$

$e^{\frac{2}{3}\pi i}, \; e^{-\frac{2}{3}\pi i}$



polar form:

$z = re^{i\theta}$

$z^3 = r^3 e^{i3\theta} = 27 e^{i0}$

$r^3 = 27 \Rightarrow r = 3$

$3\theta = 0 + 2\pi k$

$\theta = \frac{2\pi}{3} k$

$\theta = 0, \pm\frac{2\pi}{3} \; \ldots$

$z = 3e^0, \; 3e^{i\left(\frac{2\pi}{3}\right)}, \; 3e^{i\left(-\frac{2\pi}{3}\right)}$

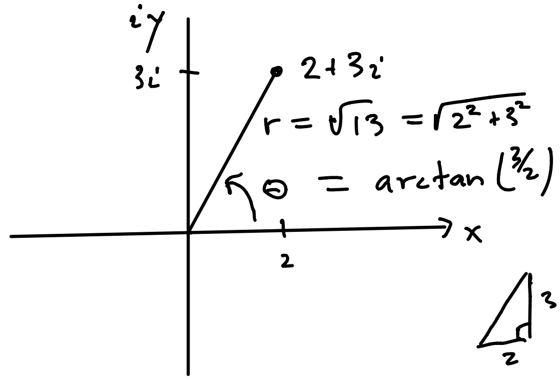
Continue with Wednesday notes + additional material on Friday.

Math 4200
Wednesday August 21
1.1-1.2 Algebra and geometry of complex arithmetic, continued.
We'll pick up in yesterday's notes where we left off, and talk about solutions to polynomial equations in today's notes.

Announcements:

usual procedure; a good idea to bring old notes

Warm-up exercise: What are the polar coords r, θ for $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$, i.e. for $2+3i \in \mathbb{C}$?



$r = \sqrt{13} = \sqrt{2^2+3^2}$

$\theta = \arctan\left(\frac{3}{2}\right) + 2\pi k$

$k \in \mathbb{Z}$