Polar form of complex numbers and the geometric meaning of complex multiplication.

Recall polar coordinates in $\mathbb{R}^2$: Every non-zero vector in $\mathbb{R}^2$ can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle from the positive $x-$ axis to the point (x, y), determined up to an integer multiple of 2π.

Translating this to complex notation and using $arg(z)$ ("argument of z") for θ:

$$z = x + i\,y = r\cos(\theta) + i\,r\sin(\theta)$$

$$|z| = \sqrt{x^2 + y^2} = r$$

$$arg(z) := \theta.$$

<u>Theorem</u>: Let $z = r\cdot(\cos(\theta) + i\sin(\theta))$ and $w = \rho\cdot(\cos(\phi) + i\sin(\phi))$ be complex numbers written in polar form. Then

$$z\,w = r\,\rho\,(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

In other words, when you multiply two complex numbers their moduli multiply and their arguments add.

Note: If you recall Euler's formula from Math 2280 then the multiplication formula reads as follows: For

$$z = r \cdot (\cos(\theta) + i \sin(\theta)) = r\, e^{i\,\theta}$$

$$w = \rho \cdot (\cos(\phi + i \sin(\phi)) = \rho\, e^{i\,\phi},$$

the product

$$z\, w = r\, \rho\, e^{i \cdot (\theta + \phi)}.$$

Use the polar form of complex numbers for the following, and compare to the rectangular coordinates computation.

Solve $z^2 = -9$ for z. Sketch.
Find all solutions to $z^3 = 27$. Sketch.

Math 4200
Wednesday August 21
1.1-1.2 Algebra and geometry of complex arithmetic, continued.
We'll pick up in yesterday's notes where we left off, and talk about solutions to polynomial equations in today's notes.

Announcements:

Warm-up exercise:

Solving polynomial equations in $\mathbb{C}$.

1) Every non-zero complex number z_0 has two square roots, i.e solutions z to

$$z^2 = z_0$$

and they are opposites.

proof 1: Use the polar form, writing $z_0 = r\,\mathrm{e}^{i\,\theta}$, $z = \rho\,\mathrm{e}^{i\,\phi}$, with $r, \rho > 0$.

proof 2: (To convince you how great polar form is) Use rectangular coordinates: Express z_0, z in terms of their real and imaginary parts,

$$z_0 = x_0 + i\,y_0$$
$$z = x + i\,y$$
$$(x + i\,y)^2 = x_0 + i\,y_0$$
$$\begin{cases} x^2 - y^2 = x_0 \\ 2\,x\,y = y_0 \end{cases}$$

Case 1: If $y_0 \neq 0$ then $x, y \neq 0$. Solve for y from the second equation and subsitute into the first:

$$x^2 - \left(\frac{y_0}{2\,x}\right)^2 = x_0$$
$$4\,x^4 - 4\,x_0 x^2 - y_0^2 = 0.$$

Use the quadratic formula for real coefficients for x^2 and throw out the negative value:

$$x^2 = \frac{4\,x_0 + \sqrt{16\,x_0^2 + 16\,y_0^2}}{8} = \frac{x_0 + \sqrt{x_0^2 + y_0^2}}{2}.$$

There are two opposite real values of x which solve this equation, with corresponding opposite values of $y = \dfrac{y_0}{2\,x}$.

Case 2: If $y_0 = 0$ it meant that $z_0 = x_0$ was real, and you already know how to find the two square roots. If $x_0 > 0$ they will be real square roots, and if $x_0 < 0$ they will be imaginary.

3) The general degree n polynomial equation

$$p(z) := z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0.$$

$$a_{n-1}, \ldots a_1, a_0 \in \mathbb{C}$$

You've been told forever that every degree n polynomial equation has n complex roots, counting multiplicty. This fact is known as "The Fundamental Theorem of Algebra." You'll learn a beautiful "elementary" proof of the fundamental theorem of algebra in this class. It's a proof by contradiction though, and except for very special polynomials there are no explicit formulas for exact solutions.....

I told you yesterday that there is a cubic formula for cubic equations. There is also a formula for the roots of 4^{th} order polynomials. The Abel-Ruffini Theorem asserts however, that there is no general formulas for the roots of degree 5 and higher polynomial equations, such that these formulas use only the algebraic operations of addition, multiplication, and taking radicals (square roots, cube roots, etc.). One of the founders of number theory, Évariste Galois, developed "Galois Theory", which explains exactly which higher degree polynomial equations can be solved using these operations. These are topics in advanced algebra courses.

4) The special polynomial equation $z^n = 1$.
Its solutions are called "the n^{th} roots of unity", and there are n of them.

Since complex multiplication in polar form reads

$$z\,w = r\,e^{i\,\theta}\,\rho\,e^{i\,\phi} = r\,\rho\,e^{i\cdot(\theta+\phi)},$$

(where $r = |z|$, $\theta = arg(z)$, $\rho = |w|$, $\phi = arg(w)$), it's easy to check via induction, that

$$z^n = r^n\,e^{i\,n\,\theta}.$$

This formula for powers is known as "DeMoivre's formula".

So, to solve $z^n = 1$, express $z = |z|\,e^{i\,\theta}$ and solve

$$|z|^n\,e^{i\,n\,\theta} = 1.$$

At the end of section 1.2, our text lists a number of identities and estimates that we'll use going forward.... we'll be doing analysis soon and will be using the triangle inequality, for example. We've already seen some of these estimates, and can discuss the ones that seem new.

Write $z = x + i\,y$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$. So $\overline{z} = x - i\,y$.

1) $z + w = \overline{z} + \overline{w}$

2) $z\overline{w} = \overline{z}\,\overline{w}$ $\qquad \left(\dfrac{z}{w}\right) = \dfrac{\overline{z}}{\overline{w}}$

3) $|z|^2 = z\,\overline{z}$

4) $z = \overline{z}$ if and only if z is real. $\qquad z = -\overline{z}$ if and only if z is imaginary.

5) $\mathrm{Re}(z) = \dfrac{1}{2}(z + \overline{z})$

$\mathrm{Im}(z) = \dfrac{1}{2\,i}(z - \overline{z})$

6) $\overline{\overline{z}} = z$

7) $|z\,w| = |z|\,|w|$ $\qquad \left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$

8) $-|z| \leq \mathrm{Re}(z) \leq |z|$ $\qquad$ i.e. $|\mathrm{Re}(z)| \leq |z|$

$-|z| \leq \mathrm{Im}(z) \leq |z|$ $\qquad$ i.e. $|\mathrm{Im}(z)| \leq |z|$

9) $|\overline{z}| = |z|$

10) $|z + w| \leq |z| + |w|$ triangle inequality

11) $|z - w| \geq |z| + |w|$ reverse triangle inequality

12) $\left|\sum_{j=1}^{n} z_j\, w_j\right|^2 \leq \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right)$ complex Cauch-Schwarz.