Welcome!

Math 4200-001 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These week we plan to cover most of sections 1.1-1.3

Monday August 19

- Go over course information on syllabus and course homepage:

  http://www.math.utah.edu/~korevaar/4200fall19

- Notice that our first homework assignment is due next Wednesday.

Then, let's begin!

Mathematics 4200-001 Fall 2019

Class Time and Place: M, W, F 11:50-12:40 LCB 222

Class website http://www.math.utah.edu/~korevaar/4200fall19

Instructor: Professor Nick Korevaar 801-581-7318
LCB 204 korevaar@math.utah.edu

Office Hours: M, W 12:50-1:30 T 11:50-12:40, and by appointment.

Text: *Basic Complex Analysis, third edition* by Jerrold E. Marsden and Michael J. Hoffman

Prerequisites: Math 3210-3220 or equivalent; we will use concepts from analysis including estimation via the triangle inequality; continuity; the derivative matrix and differentiability of multivariable functions; path integrals and Green's Theorem. Section 1.4 of the text contains a review of many but not all of these concepts, as they will be applied in our context. (We will spend about two lectures on this review.) You will be expected to learn and be able to explain the key theorems in this course, and your homework will include theoretical problems along with computations and applications.

Course Description: The unfortunately named ``imaginary" and ``complex numbers" were originally introduced by Geronimo Cardano in the 1500's as an algebraic artifice to factor polynomials. Probably all of you first encountered *i*, the square root of -1, and complex numbers $a+bi$ when factoring quadratic equations. You may know, e.g. from Math 2270, that complex numbers share the same field axioms for addition and multiplication as do the real numbers. You have also seen Leonhard Euler's beautiful formula from the 1600's,

$$e^{i\theta} = cos(\theta)+i\ sin(\theta),$$

and its applications to differential equations. However, it was not until the 1800's that mathematicians including Karl Friedrich Gauss, Augustin Cauchy, Peter Dirichlet, Karl Weierstrass and Georg Friedrich Bernhard Riemann more fully developed the field known as Complex Analysis. This is a core area of study and to the present day remains an essential tool in many areas of mathematics and science.

In this class we will systematically develop the theory, the calculus and the magic of complex analysis, chapters 1-5 of our text. In chapters 5 and 8 we will see some classical applications of complex analysis to partial and ordinary differential equations. Time permitting, and hopefully with student project input, we will see other diverse applications of complex analysis, for example, to fluid mechanics, minimal surfaces, Riemann surfaces, Julia set fractals, hyperbolic geometry, the prime number theorem, or other suitable topics which suit your fancy and are also agreeable to me.

Grading: There will be two midterms, a comprehensive final examination, and homework. Each midterm will count for 20% of your grade, homework will count for 30%, and the final exam will make up the remaining 30%. All exams will be given in our classroom. The midterm exam dates are **Wednesday October 2** and **Wednesday November 13**. The final exam is at the University time and date of **Friday December 13, 10:30-12:30** in our usual classroom.

You may opt out of the final exam by completing a project (by yourself or with one or two other people) on some application as indicated above. Each project shall consist of a 5-10 page expository paper, and a presentation to the class of at least 20 minutes in length, but possibly longer. I will be available for pre-presentation consultation and practice. Project groups and topics must be approved by me, **by Friday November 1**.

Homework assigned by Wednesday of each week will be collected the following Wednesday, in order that it may be graded. Work individually and collaboratively, but everyone should carefully write up their own final solutions to hand in.

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Math 4200-001
Week 1 concepts and homework
1.1-1.3
Due Wednesday August 28. at start of class.

1) On the history of complex numbers: There is a Coursera class on complex analysis. The introductory lecture, which is 19 minutes of video with some pauses for you to work, goes into the history of their discovery. Although you first saw the "square root of -1" in the quadratic formula, i was actually introduced to find real number solutions to cubic equations, because in the real world in the 1600's, people were only interested in "real" solutions! And that's why i was called imaginary. (Every cubic has at least one real root, by the intermediate value theorem.) The class is taught by Dr. Petra Bonfert-Taylor and I really like this first lecture. Find the internet link, watch and work along. Nothing to hand in here, but you might decide you want to audit the class as a supplement to ours, and you can do that for free. The Coursera class runs from now until October 14.

Text problems: There are example exercises at the end of each section which I recommend looking over before you try the homework. Odd answers are in the back of the text.

1.1 1b, 2c, 4ab, 6a, 10, 11 (just check the multiplication axioms and the distributive law), 14, 17a.

1.2 1a, 2b, 4, 5, 8 (note, "absolute value" means "modulus"), 11, 14, 19.

1.3 1a, 4b, 5a, 6a, 7a, 8a, 10, 21, 23, 30b.

w1.1 Sketch the following subsets of the complex plane. Use shading and labeling to clearly specify the subset.
a) $\{z \in \mathbb{C} \mid 1 \leq \mathrm{Re}(z) < 3\,, 0 < \mathrm{Im}(z) < 2\}$.
b) $\left\{z \in \mathbb{C} \,\middle|\, |z| \leq 2, 0 < arg(z) < \frac{\pi}{2}\right\}$
c) The image of the sector in b), under the transformation $f(z) = z^3$.

w1.2 Sketch the following subsets of the complex plane, as above.
a) $\left\{z \in \mathbb{C} \,\middle|\, -\frac{\pi}{4} \leq \mathrm{Im}(z) \leq \frac{\pi}{4}\right\}$
b) The image of the strip in a), under the transformation $f(z) = e^z$.
c) The image of the right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ under the transformation $g(z) = \log(z)$, where $arg(1)$ is chosen to be 2π.

Complex analysis is like Calculus - it's based on derivatives and integrals - except that the functions $f(z)$ have complex number domains and ranges, i.e. subsets of the *complex plane*. For example, the limit definition of derivative looks just like it did in calculus, but the function inputs are complex numbers. This changes the character of the theory in magical ways from that of regular Calculus.

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad z, h \in \mathbb{C}$$

You may or may not have discussed the complex plane in a linear algebra course, since it is isomorphic to the real vector space $\mathbb{R}^2$. It's a good place to start this course.

<u>Definition</u> The "algebra" of *complex numbers* $\mathbb{C}$ is defined as the set

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}$$

together with the operations of addition and scalar multiplication defined by

$$(x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1 x_2 - y_1 y_2) + i(\boxed{x_1 y_2} + \boxed{y_1 x_2}),$$
$$\text{for all } x_1, y_1, x_2, y_2 \in \mathbb{R}.$$

The definition for complex multiplication follows from the usual axioms for real number multiplication and the definition that $i^2 := -1$.

$$x_1(x_2 + i y_2) + i y_1 (x_2 + i y_2)$$
$$= x_1 x_2 + \boxed{x_1 i y_2} + \boxed{i y_1 x_2} + i y_1 i y_2$$

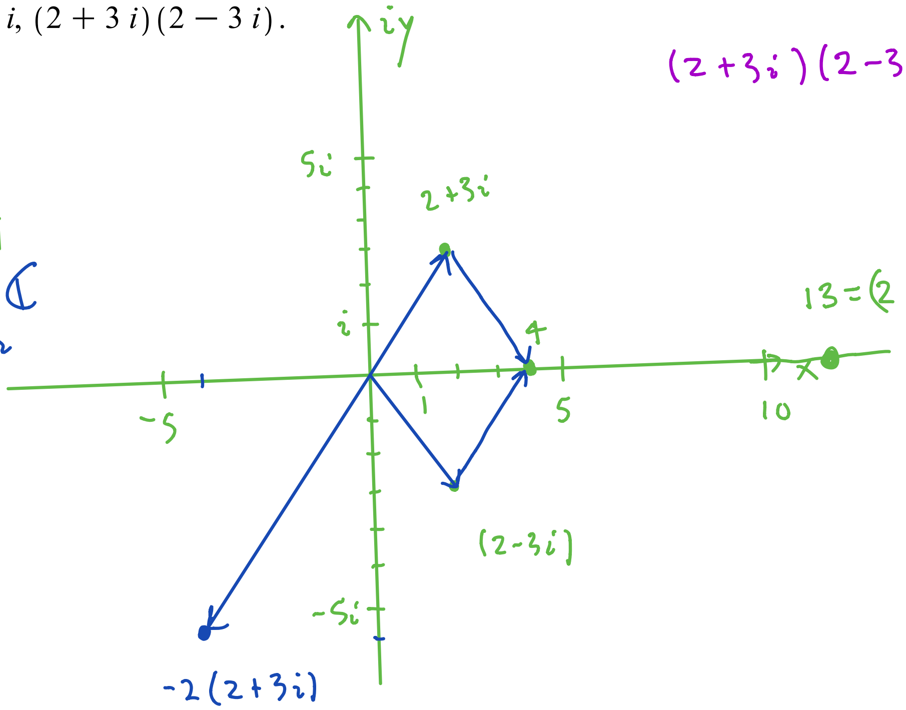
It is natural to identify each complex number $x + i\,y$ with the corresponding point $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. This identification and the usual representation for $\mathbb{R}^2$ is how we define the *complex plane* $\mathbb{C}$.

Sketch some points in the complex plane and add and multiply some complex numbers e.g.
$2 + 3\,i,\ -2(2 + 3\,i),\ 2 - 3\,i,\ (2 + 3\,i)(2 - 3\,i)$.

$\|$
$-4 - 6i$

$(2+3i) + (2-3i) = 4$

complex addition in $\mathbb{C}$ corresponds to vector addition in $\mathbb{R}^2$



$-9i^2$

$(2+3i)(2-3i) = 4 + 9 = 13$
$+ 0i$

$6i - 6i$

$13 = (2+3i)(2-3i)$?!

Under this identification of $\mathbb{C}$ with $\mathbb{R}^2$, the definition for complex number addition just corresponds to vector addition in $\mathbb{R}^2$, which we understand. The product of a real number with a complex number corresponds to scalar multiplication in $\mathbb{R}^2$, which we also understand.

The formula for complex multiplication also has geometric meaning when we consider the corresponding points in $\mathbb{R}^2$, and it's more interesting than just vector addition and scalar multiplication, as we'll see today.

Complex number addition corresponds to vector addition in $\mathbb{R}^2$:

$$(x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2)$$
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix};$$

Complex number multiplication interpreted as a strange operation in $\mathbb{R}^2$:

$$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1\,x_2 - y_1\,y_2 \\ x_1\,y_2 + y_1\,x_2 \end{bmatrix}.$$

weird,

unless $x_1(x_2 + iy_2) = x_1x_2 + i\,x_1y_2$

like scalar mult by x_1, in $\mathbb{R}^2$

$\mathbb{C}$ satisfies the *field axioms* of Algebra: Let $z, w, s \in \mathbb{C}$. Then we have the following properties:

the addition axioms, which correspond to vector space axioms in $\mathbb{R}^2$:

$$z + w = w + z$$

$$z + (w + s) = (z + w) + s$$

$$z + 0 = z$$

$$z + (-1 \cdot z) = 0;$$

✓

the multiplication axioms, some of which you will check in homework:

$z = x + iy$
$w = u + iv$ $\quad x, y, u, v \in \mathbb{R}$

$$z\,w = w\,z$$

e.g.

$zw = (x+iy)(u+iv)$
$wz = (u+iv)(x+iy)$

expand, verify $zw = wz$

$$(z\,w)\,s = z(w\,s)$$

$$1\,z = z$$

each $z \neq 0$ has unique z^{-1} which we write as $\frac{1}{z}$, such that $z\,z^{-1} = 1$;

one mult. inv.

(only 1): if $\begin{cases} z(z_1) = 1 \\ z(z_2) = 1 \end{cases}$

then $z_2(\underbrace{z z_1}_{1}) = z_2$

$\quad \| $

$(\underbrace{z_2 z}_{1}) z_1 = z_1$

distributive property:

$$z(w + s) = z\,w + z\,s.$$

$z = 2 + 3i$

$\frac{1}{z} = \frac{1}{2+3i} = a + ib$

$= \frac{1}{2+3i}\,\frac{2-3i}{2-3i} = \frac{2-3i}{13}$

$(2+3i)\frac{1}{13}(2-3i) = \frac{1}{13} \cdot 13 = 1$ ✓

Important operations for complex numbers:

Let $z = x + i\,y$ with $x, y \in \mathbb{R}$. Then

$Re(z) := x$ "Real part of z"

$Im(z) := y$ "Imaginary part of z" (even though it's a real number)

$\bar{z} := x - i\,y$ "conjugate of z" or "z bar".

$|z| := \sqrt{x^2 + y^2}$ "modulus of z" or "magnitude of z"

text says "absolute value"

$Re(2+3i) = 2$

$Im(2+3i) = 3$

$\overline{2+3i} = 2-3i$

$|2+3i| = \sqrt{13}$

corresponds to vector magnitudes

Check:

$\overline{zw} = \bar{z}\,\bar{w}$.

$\overline{(x+iy)(u+iv)} = \overline{(xu-yv)+i(xv+yu)} = (xu-yv) - i(xv+yu)$

$\bar{z}\bar{w} = (x-iy)(u-iv) = xu - yv + i(-xv-yv)$

$\uparrow$

$(-iy)(-iv)$

$-1\,yv$

$|z|^2 = z\,\bar{z}$ so $|z| = \sqrt{z\,\bar{z}}$.

- $|zw| = |z||w|$

$z^{-1} = \dfrac{\bar{z}}{|z|^2}$

Compare to our earlier example, where $z = 2 + 3\,i$.