Math 4200-001 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These week we plan to cover most of sections 1.1-1.3

Monday August 19

- Go over course information on syllabus and course homepage:

  http://www.math.utah.edu/~korevaar/4200fall19

- Notice that our first homework assignment is due next Wednesday.

Then, let's begin!

Complex analysis is like Calculus - it's based on derivatives and integrals - except that the functions $f(z)$ have complex number domains and ranges, i.e. subsets of the *complex plane*. For example, the limit definition of derivative looks just like it did in calculus, but the function inputs are complex numbers. This changes the character of the theory in magical ways from that of regular Calculus.

You may or may not have discussed the complex plane in a linear algebra course, since it is isomorphic to the real vector space $\mathbb{R}^2$. It's a good place to start this course.

<u>Definition</u> The "algebra" of *complex numbers* $\mathbb{C}$ is defined as the set

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}$$

together with the operations of addition and scalar multiplication defined by

$$\left(x_1 + i\,y_1\right) + \left(x_2 + i\,y_2\right) := \left(x_1 + x_2\right) + i\left(y_1 + y_2\right)$$
$$\left(x_1 + i\,y_1\right)\left(x_2 + i\,y_2\right) := \left(x_1 x_2 - y_1 y_2\right) + i\left(x_1 y_2 + y_1 x_2\right),$$
$$\text{for all } x_1, y_1, x_2, y_2 \in \mathbb{R}.$$

The definition for complex multiplication follows from the usual axioms for real number multiplication and the definition that $i^2 := -1$.

It is natural to identify each complex number $x + i\,y$ with the corresponding point $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. This identification and the usual representation for $\mathbb{R}^2$ is how we define the *complex plane* $\mathbb{C}$.

Sketch some points in the complex plane and add and multiply some complex numbers e.g. $2 + 3\,i,\ -2(2 + 3\,i),\ 2 - 3\,i,\ (2 + 3\,i)(2 - 3\,i)$.

Under this identification of $\mathbb{C}$ with $\mathbb{R}^2$, the definition for complex number addition just corresponds to vector addition in $\mathbb{R}^2$, which we understand. The product of a real number with a complex number corresponds to scalar multiplication in $\mathbb{R}^2$, which we also understand.

The formula for complex multiplication also has geometric meaning when we consider the corresponding points in $\mathbb{R}^2$, and it's more interesting than just vector addition and scalar multiplication, as we'll see today.

Complex number addition corresponds to vector addition in $\mathbb{R}^2$:

$$(x_1 + i y_1) + (x_2 + i y_2) := (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix};$$

Complex number multiplication interpreted as a strange operation in $\mathbb{R}^2$:

$$(x_1 + i y_1)(x_2 + i y_2) := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + y_1 x_2 \end{bmatrix}.$$

$\mathbb{C}$ satisfies the *field axioms* of Algebra: Let $z, w, s \in \mathbb{C}$. Then we have the following properties:

the addition axioms, which correspond to vector space axioms in $\mathbb{R}^2$:

$$z + w = w + z$$

$$z + (w + s) = (z + w) + s$$

$$z + 0 = z$$

$$z + (-1 \cdot z) = 0;$$

the multiplication axioms, some of which you will check in homework:

$$z\,w = w\,z$$

$$(z\,w)\,s = z(w\,s)$$

$$1\,z = z$$

each $z \neq 0$ has unique z^{-1} which we write as $\dfrac{1}{z}$, such that $z\,z^{-1} = 1$;

distributive property:

$$z(w + s) = z\,w + z\,s.$$

Important operations for complex numbers:

Let $z = x + i\,y$ with $x, y \in \mathbb{R}$. Then

$Re(z) := x$ "Real part of z"

$Im(z) := y$ "Imaginary part of z" (even though it's a real number)

$\overline{z} := x - i\,y$ "conjugate of z" or "z bar".

$|z| := \sqrt{x^2 + y^2}$ "modulus of z" or "magnitude of z"

Check:

$\overline{zw} = \overline{z}\,\overline{w}$.

$|z|^2 = z\,\overline{z}$ so $|z| = \sqrt{z\,\overline{z}}$.

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

Compare to our earlier example, where $z = 2 + 3\,i$.

Polar form of complex numbers and the geometric meaning of complex multiplication.

Recall polar coordinates in $\mathbb{R}^2$: Every non-zero vector in $\mathbb{R}^2$ can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle from the positive $x-$ axis to the point (x, y), determined up to an integer multiple of 2π.

Translating this to complex notation and using $arg(z)$ ("argument of z") for θ:

$$z = x + i\,y = r\cos(\theta) + i\,r\sin(\theta)$$

$$|z| = \sqrt{x^2 + y^2} = r$$

$$arg(z) := \theta.$$

<u>Theorem</u>: Let $z = r\cdot(\cos(\theta) + i\sin(\theta))$ and $w = \rho\cdot(\cos(\phi) + i\sin(\phi))$ be complex numbers written in polar form. Then

$$z\,w = r\,\rho\,(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

In other words, when you multiply two complex numbers their moduli multiply and their arguments add.

Note: If you recall Euler's formula from Math 2280 then the multiplication formula reads as follows: For

$$z = r\cdot\left(\cos(\theta) + i\sin(\theta)\right) = r\,e^{i\,\theta}$$

$$w = \rho\cdot\left(\cos(\phi + i\sin(\phi)\right) = \rho\,e^{i\,\phi},$$

the product

$$z\,w = r\,\rho\,e^{i\cdot(\theta + \phi)}.$$

Use the polar form of complex numbers for the following, and compare to the rectangular coordinates computation.

Solve $z^2 = -9$ for z. Sketch.
Find all solutions to $z^3 = 27$. Sketch.