Chapter 1

Analytic Functions

In this chapter the basic ideas about complex numbers and analytic functions are introduced. The organization of the text is analogous to that of an elementary calculus textbook, which begins by introducing $\mathbb{R}$, the set of real numbers, and functions $f(x)$ of a real variable x. One then studies the theory and practice of differentiation and integration of functions of a real variable. Similarly, in complex analysis we begin by introducing $\mathbb{C}$, the set of complex numbers z. We then study functions $f(z)$ of a complex variable z, which are differentiable in a complex sense; these are called analytic functions.

The analogy between real and complex variables is, however, a little deceptive, because complex analysis is a surprisingly richer theory; a lot more can be said about an analytic function than about a differentiable function of a real variable, as will be fully developed in subsequent chapters.

In addition to becoming familiar with the theory, the student should strive to gain facility with the standard (or "elementary") functions—such as polynomials, e^z, $\log z$, $\sin z$—as in calculus. These functions are studied in §1.3 and appear frequently throughout the text.

1.1 Introduction to Complex Numbers

The following discussion will assume some familiarity with the main properties of real numbers. The real number system resulted from the search for a system (an abstract set together with certain rules) that included the rationals but that also provided solutions to such polynomial equations as $x^2 - 2 = 0$.

Historical Perspective Historically, a similar consideration gave rise to an extension of the real numbers. As early as the sixteenth century, Geronimo Cardano considered quadratic (and cubic) equations such as $x^2 + 2x + 2 = 0$, which is satisfied by no real number x. The quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$ yields "formal" expressions for the two solutions of the equation $ax^2 + bx + c = 0$. But this formula may involve square roots of negative numbers; for example, $-1 \pm \sqrt{-1}$ for the equation $x^2 + 2x + 2 = 0$. Cardano noticed that if these "complex numbers" were treated as ordinary numbers with the added rule that $\sqrt{-1} \cdot \sqrt{-1} = -1$, they did indeed solve the equations.

The important expression $\sqrt{-1}$ is now given the widely accepted designation $i = \sqrt{-1}$. (An alternative convention is followed by many electrical engineers, who prefer the symbol $j = \sqrt{-1}$ since they wish to reserve the symbol i for electric current.) However, in the past it was felt that no meaning could actually be assigned to such expressions, which were therefore termed "imaginary." Gradually, especially as a result of the work of Leonhard Euler in the eighteenth century, these imaginary quantities came to play an important role. For example, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ revealed the existence of a profound relationship between complex numbers and the trigonometric functions. The rule $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$ was found to summarize the rules for expanding sine and cosine of a sum of two angles in a neat way, and this result alone indicated that some meaning should be attached to these "imaginary" numbers.

However, it took nearly three hundred years until the work of Casper Wessel (ca. 1797), Jean Robert Argand (1806), Karl Friedrich Gauss (1831), Sir William R. Hamilton (1837), and others, when "imaginary" numbers were recognized as legitimate mathematical objects, and it was realized that there is nothing "imaginary" about them at all (although this term is still used).

The complex analysis that is the subject of this book was developed in the nineteenth century, mainly by Augustin Cauchy (1789–1857). Later his theory was made more rigorous and extended by such mathematicians as Peter Dirichlet (1805–1859), Karl Weierstrass (1815–1897), and Georg Friedrich Bernhard Riemann (1826–1866).

The search for a method to describe heat conduction influenced the development of the theory, which has found many uses outside mathematics. Subsequent chapters will discuss some of these applications to problems in physics and engineering, such as hydrodynamics and electrostatics. The theory also has mathematical applications to problems that at first do not seem to involve complex numbers. For example, the proof that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2},$$

or that

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin(a\pi)},$$

(where $0 < a < 1$), or that

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \frac{2\pi}{\sqrt{a^2-1}},$$

may be difficult or, in some cases, impossible using elementary calculus, but these identities can be readily proved using the techniques of complex variables.

The Complex Number System Complex analysis has become an indispensable and standard tool of the working mathematician, physicist, and engineer. Neglect of it can prove to be a severe handicap in most areas of research and application involving mathematical ideas and techniques. The first objective of this section will be to define complex numbers and to show that the usual algebraic manipulations hold. To begin, recall that the xy plane, denoted by $\mathbb{R}^2$, consists of all ordered pairs (x, y) of real numbers.

Definition 1.1.1 *The* ***system of complex numbers****, denoted $\mathbb{C}$, is the set $\mathbb{R}^2$ together with the usual rules of* ***vector addition*** *and* ***scalar multiplication*** *by a real number a, namely,*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ a(x, y) &= (ax, ay)\end{aligned}$$

and with the operation of ***complex multiplication****, defined by*

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

We will need to explain where this strange rule of multiplication comes from! Rather than using (x, y) to represent a complex number, we will find it more convenient to return to more standard notation as follows. Let us identify *real* numbers x with points on the x axis; thus x and $(x, 0)$ stand for the same point $(x, 0)$ in $\mathbb{R}^2$. The y axis will be called the ***imaginary axis***, and the unit point $(0, 1)$ will be denoted i. Thus, by definition, $i = (0, 1)$. Then

$$(x, y) = x + yi$$

because the right side of the equation stands for

$$(x, 0) + y(0, 1) = (x, 0) + (0, y) = (x, y).$$

Using $y = (y, 0)$ and Definition 1.1.1 of complex multiplication, we get

$$iy = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, y \cdot 1 + 0 \cdot 0) = (0, y) = y(0, 1) = yi,$$

so we can also write $(x, y) = x + iy$. A single symbol such as $z = a + ib$ is generally used to indicate a complex number. The notation $z \in \mathbb{C}$ means that z belongs to the set of complex numbers.

Note that

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, (1 \cdot 0 + 0 \cdot 1)) = (-1, 0) = -1,$$

so we do have the property we want:

$$i^2 = -1.$$

If we remember this equation, then the rule for multiplication of complex numbers is also easy to remember and motivate:

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2bd \\ &= (ac-bd) + i(ad+bc).\end{aligned}$$

For example, $2+3i$ is the complex number $(2,3)$, and

$$(2+3i)(1-4i) = 2 - 12i^2 + 3i - 8i = 14 - 5i$$

is another way of saying that

$$(2,3)(1,-4) = (2\cdot 1 - 3(-4), 3\cdot 1 + 2(-4)) = (14,-5).$$

The reason for using the expression $a+bi$ is twofold. First, it is conventional. Second, the rule $i^2=-1$ is easier to use than the rule $(a,b)(c,d) = (ac-bd, bc+ad)$, although both rules produce the same result.

Because multiplication of real numbers is associative, commutative, and distributive, it is reasonable to expect that multiplication of complex numbers is also; that is, for all complex numbers z, w, and s we have

$$(zw)s = z(ws), \quad zw = wz, \quad \text{and} \quad z(w+s) = zw + zs.$$

Let us verify the first of these properties; the others can be similarly verified.

Let $z = a+ib$, $w = c+id$, and $s = e+if$. Then $zw = (ac-bd) + i(bc+ad)$, so

$$(zw)s = e(ac-bd) - f(bc+ad) + i[e(bc+ad) + f(ac-bd)].$$

Similarly,

$$\begin{aligned}z(ws) &= (a+bi)[(ce-df) + i(cf+de)] \\ &= a(ce-df) - b(cf+de) + i[a(cf+de) + b(ce-df)].\end{aligned}$$

Comparing these expressions and accepting the usual properties of real numbers, we conclude that $(zw)s = z(ws)$. Thus we can write, without ambiguity, an expression like $z^n = z\cdot \ldots \cdot z$ (n times).

Note that $a+ib = c+id$ means $a=c$ and $b=d$ (since this is what equality means in $\mathbb{R}^2$) and that 0 stands for $0+i0 = (0,0)$. Thus $a+ib=0$ means that *both* $a=0$ and $b=0$.

In what sense are these complex numbers an extension of the reals? We have already said that if a is real we also write a to stand for $a+0i = (a,0)$. In other words, the reals $\mathbb{R}$ are identified with the x axis in $\mathbb{C} = \mathbb{R}^2$; we are thus regarding the real numbers as those complex numbers $a+bi$ for which $b=0$. If, in the expression $a+bi$, the term $a=0$, we call $bi = 0+bi$ a ***pure imaginary number***. In the expression $a+bi$ we say that a is the ***real part*** and b is the ***imaginary part***. This is sometimes written $\operatorname{Re} z = a$, $\operatorname{Im} z = b$, where $z = a+bi$. Note that $\operatorname{Re} z$ and $\operatorname{Im} z$ are always real numbers (see Figure 1.1.1).

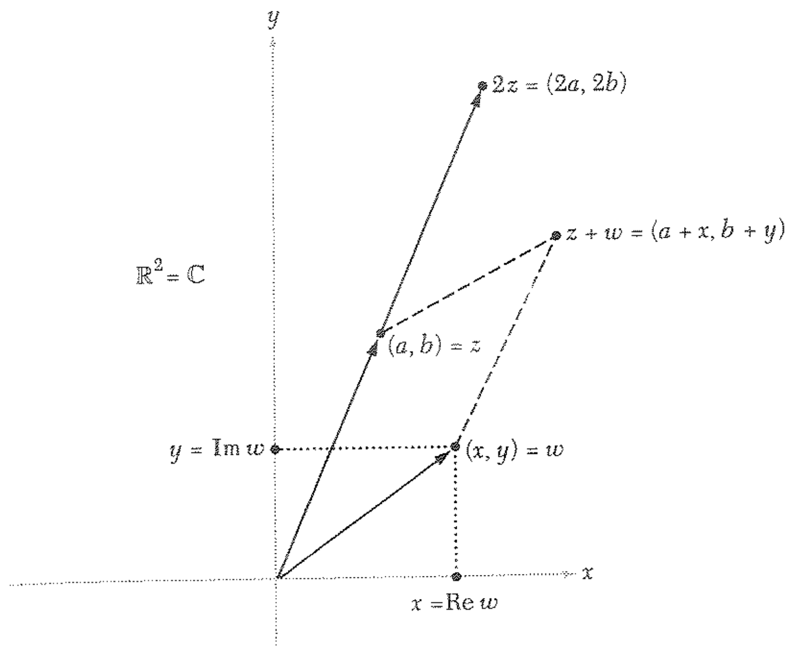

Figure 1.1.1: The geometry of complex numbers.

Algebraic Properties The complex numbers obey all the algebraic rules that ordinary real numbers do. For example, it will be shown in the following discussion that multiplicative inverses exist for nonzero elements. This means that if $z \neq 0$, then there is a (complex) number z' such that $zz' = 1$, and we write $z' = z^{-1}$. We can write this expression unambiguously (in other words, z' is uniquely determined), because if $zz'' = 1$ as well, then $z' = z' \cdot 1 = z'(zz'') = (z'z)z'' = 1 \cdot z'' = z''$, and so $z'' = z'$. To show that z' exists, suppose that $z = a + ib \neq 0$. Then at least one of $a \neq 0, b \neq 0$ holds, and so $a^2 + b^2 \neq 0$. To find z', we set $z' = a' + b'i$. The condition $zz' = 1$ imposes conditions that will enable us to compute a' and b'. Computing the product gives $zz' = (aa' - bb') + (ab' + a'b)i$. The linear equations $aa' - bb' = 1$ and $ab' + a'b = 0$ can be solved for a' and b' giving $a' = a/(a^2 + b^2)$ and $b' = -b/(a^2 + b^2)$, since $a^2 + b^2 \neq 0$. Thus for $z = a + ib \neq 0$, we may write

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}.$$

Having found this candidate for z^{-1} it is now a straightforward, albeit tedious, computation to check that it works.

If z and w are complex numbers with $w \neq 0$, then the symbol z/w means zw^{-1}; we call z/w the ***quotient*** of z by w. Thus $z^{-1} = 1/z$. To compute z^{-1}, the following series of equations is common and is a useful way to remember the preceding formula for z^{-1}:

$$\frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

In short, all the usual algebraic rules for manipulating real numbers, fractions, polynomials, and so on, hold for complex numbers.

Formally, the system of complex numbers is an example of a ***field***. The crucial rules for a field, stated here for reference, are

Addition rules

(i) $z + w = w + z$

(ii) $z + (w + s) = (z + w) + s$

(iii) $z + 0 = z$

(iv) $z + (-z) = 0$

Multiplication rules

(i) $zw = wz$

(ii) $(zw)s = z(ws)$

(iii) $1z = z$

(iv) $z(z^{-1}) = 1$ for $z \neq 0$

Distributive law $z(w + s) = zw + zs$

In summary, we have

Theorem 1.1.2 *The complex numbers $\mathbb{C}$ form a field.*

The student is cautioned that we generally do not define inequalities like $z \leq w$, for complex z and w. If one requires the usual ordering properties for reals to hold, then *such an ordering is impossible* for complex numbers.[1] Thus in this text the notation $z \leq w$ will be avoided unless z and w happen to be real.

Roots of Quadratic Equations As mentioned previously, one of the reasons for using complex numbers is to enable us to take square roots of negative real numbers. That this can, in fact, be done for all complex numbers is verified in the next proposition.

Proposition 1.1.3 *Let $z \in \mathbb{C}$. Then there exists a complex number $w \in \mathbb{C}$ such that $w^2 = z$. (Notice that $-w$ also satisfies this equation.)*

[1]This statement can be proved as follows. Suppose that such an ordering exists. Then either $i \geq 0$ or $i \leq 0$. Suppose that $i \geq 0$. Then $i \cdot i \geq 0$, so $-1 \geq 0$, which is absurd. Alternatively, suppose that $i \leq 0$. Then $-i \geq 0$, so $(-i)(-i) \geq 0$, that is, $-1 \geq 0$, again absurd. If $z = a + ib$ and $w = c + id$, we could say that $z \leq w$ iff $a \leq c$ and $b \leq d$. This is an ordering of sorts, but it does not satisfy all the rules that might be required, such as those obeyed by real numbers.

Proof (We shall give a purely algebraic proof here; another proof, based on polar coordinates, is given in §1.2.) Let $z = a + bi$. We want to find $w = x + iy$ such that $z = w^2$; *i.e.*, $a + bi = (x + iy)^2 = (x^2 - y^2) + (2xy)i$, and so we must simultaneously solve $x^2 - y^2 = a$ and $2xy = b$. The existence of such solutions is geometrically clear from examination of the graphs of the two equations. These graphs are shown in Figure 1.1.2 for the case in which both a and b are positive. From the graphs it is clear that there should be two solutions which are negatives of each other. In the following paragraph, these will be obtained algebraically.

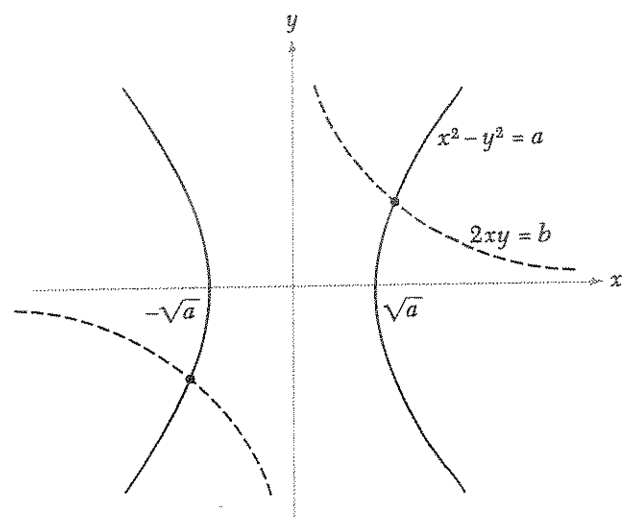

Figure 1.1.2: Graphs of the curves $x^2 - y^2 = a$ and $2xy = b$.

We know that $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2$. Hence $x^2 + y^2 = \sqrt{a^2 + b^2}$, so $x^2 = \left(a + \sqrt{a^2 + b^2}\right)/2$ and $y^2 = \left(-a + \sqrt{a^2 + b^2}\right)/2$. If we let

$$\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}},$$

where $\sqrt{\ }$ denotes the positive square root of positive real numbers, then, in the event that b is positive, we have either $x = \alpha, y = \beta$ or $x = -\alpha, y = -\beta$; in the event that b is negative, we have either $x = \alpha, y = -\beta$ or $x = -\alpha, y = \beta$. We conclude that the equation $w^2 = z$ has solutions $\pm(\alpha + \mu\beta i)$, where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$. ■

The formula for square roots developed in this proof is worth summarizing explicitly. Namely, *the two (complex) square roots of* $a + ib$ *are given by*

$$\sqrt{a + ib} = \pm(\alpha + \mu\beta i),$$

where α *and* β *are given by the displayed formula preceding this one and where* $\mu = 1$ *if* $b \geq 0$ *and* $\mu = -1$ *if* $b < 0$. From the expressions for α and β we can conclude three things:

1. The square roots of a complex number are real if and only if the complex number is real and positive.
2. The square roots of a complex number are purely imaginary if and only if the complex number is real and negative.
3. The two square roots of a number coincide if and only if the complex number is zero.

(The student should check these conclusions.)

We can easily check that the quadratic equation $az^2 + bz + c = 0$ for complex numbers a, b, c has solutions $z = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$, where now the square root denotes the two square roots just constructed.

Worked Examples

Example 1.1.4 *Prove that $1/i = -i$ and that $1/(i+1) = (1-i)/2$.*

Solution First,

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$$

because $i \cdot -i = -(i^2) = -(-1) = 1$. Also,

$$\frac{1}{i+1} = \frac{1}{i+1}\frac{1-i}{1-i} = \frac{1-i}{2}$$

since $(1+i)(1-i) = 1+1 = 2$.

Example 1.1.5 *Find the real and imaginary parts of $(z+2)/(z-1)$ where $z = x+iy$.*

Solution We start by writing the fraction in terms of the real and imaginary parts of z and "rationalizing the denominator". Namely,

$$\begin{aligned} \frac{z+2}{z-1} = \frac{(x+2)+iy}{(x-1)+iy} &= \frac{(x+2)+iy}{(x-1)+iy} \cdot \frac{(x-1)-iy}{(x-1)-iy} \\ &= \frac{(x+2)(x-1)+y^2+i[y(x-1)-y(x+2)]}{(x-1)^2+y^2}. \end{aligned}$$

Hence,

$$\operatorname{Re}\frac{z+2}{z-1} = \frac{x^2+x-2+y^2}{(x-1)^2+y^2}$$

and

$$\operatorname{Im}\frac{z+2}{z-1} = \frac{-3y}{(x-1)^2+y^2}.$$

Example 1.1.6 *Solve the equation $z^4 + i = 0$ for z.*

Solution Let $z^2 = w$. Then the equation becomes $w^2 + i = 0$. Now we use the formula $\sqrt{a+ib} = \pm(\alpha + \mu\beta i)$ we developed for taking square roots. Letting $a = 0$ and $b = -1$, we get

$$w = \pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right).$$

Consider the equation $z^2 = (1-i)/\sqrt{2}$. Using the same formula for square roots, but now letting $a = 1/\sqrt{2}$ and $b = -1/\sqrt{2}$, we obtain the two solutions

$$z = \pm\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i\right).$$

From the second possible value for w we obtain two further solutions:

$$z = \pm\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2+\sqrt{2}}}{2}i\right).$$

In the next section, de Moivre's formula will be developed, which will enable us to find the nth root of any complex number rather simply.

Example 1.1.7 *Prove that, for complex numbers z and w,*

$$\operatorname{Re}(z+w) = \operatorname{Re} z + \operatorname{Re} w$$

and

$$\operatorname{Im}(z+w) = \operatorname{Im} z + \operatorname{Im} w.$$

Solution Let $z = x + iy$ and $w = a + ib$. Then $z + w = (x+a) + i(y+b)$, so $\operatorname{Re}(z+w) = x + a = \operatorname{Re} z + \operatorname{Re} w$. Similarly, $\operatorname{Im}(z+w) = y + b = \operatorname{Im} z + \operatorname{Im} w$.

Exercises

1. Express the following complex numbers in the form $a + ib$:

   (a) $(2+3i) + (4+i)$

   (b) $\dfrac{2+3i}{4+i}$

   (c) $\dfrac{1}{i} + \dfrac{3}{1+i}$

2. Express the following complex numbers in the form $a + bi$:

   (a) $(2+3i)(4+i)$

   (b) $(8+6i)^2$

(c) $\left(1+\frac{3}{1+i}\right)^2$

3. Find the solutions to $z^2 = 3 - 4i$.

4. Find the solutions to the following equations:

   (a) $(z+1)^2 = 3 + 4i$

   (b) $z^4 - i = 0$

5. Find the real and imaginary parts of the following, where $z = x + iy$:

   (a) $\frac{1}{z^2}$

   (b) $\frac{1}{3z+2}$

6. Find the real and imaginary parts of the following, where $z = x + iy$:

   (a) $\frac{z+1}{2z-5}$

   (b) z^3

7. Is it true that $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$?

8. If a is real and z is complex, prove that $\operatorname{Re}(az) = a \operatorname{Re} z$ and $\operatorname{Im}(az) = a \operatorname{Im} z$. Generally, show that $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ is a real linear map; that is, $\operatorname{Re}(az + bw) = a \operatorname{Re} z + b \operatorname{Re} w$ for a, b real and z, w complex.

9.• Show that $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ and that $\operatorname{Im}(iz) = \operatorname{Re}(z)$ for any complex number z.

10. (a) Fix a complex number $z = x + iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0), (0,1)$ of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

   (b) Show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

11. Assuming that they work for real numbers, show that the nine rules given for a field also work for complex numbers.

12. Using only the axioms for a field, give a formal proof (including all details) for the following:

   (a) $\frac{1}{z_1 z_2} = \frac{1}{z_1} \cdot \frac{1}{z_2}$

(b) $\frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1 + z_2}{z_1 z_2}$

13.* Let $(x - iy)/(x + iy) = a + ib$. Prove that $a^2 + b^2 = 1$.

14. Prove the binomial theorem for complex numbers; that is, letting z, w be complex numbers and n be a positive integer,

$$(z + w)^n = z^n + \binom{n}{1} z^{n-1} w + \binom{n}{2} z^{n-2} w^2 + \ldots + \binom{n}{n} w^n,$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Use induction on n.

15. Show that z is real if and only if $\operatorname{Re} z = z$.

16. Prove that, for each integer k,

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i.$$

Show how this result gives a formula for i^n for all n by writing $n = 4k + j, 0 \leq j \leq 3$.

17. Simplify the following:

(a) $(1 + i)^4$
(b) $(-i)^{-1}$

18. Simplify the following:

(a) $(1 - i)^{-1}$
(b) $\frac{1+i}{1-i}$

19. Simplify the following:

(a) $\sqrt{1 + \sqrt{i}}$
(b) $\sqrt{1 + i}$
(c)* $\sqrt{\sqrt{-i}}$

20. Show that the following rules uniquely determine complex multiplication on $\mathbb{C} = \mathbb{R}^2$:

(a) $(z_1 + z_2)w = z_1 w + z_2 w$
(b) $z_1 z_2 = z_2 z_1$
(c) $i \cdot i = -1$
(d) $z_1(z_2 z_3) = (z_1 z_2) z_3$
(e) If z_1 and z_2 are real, $z_1 \cdot z_2$ is the usual product of real numbers.

1.2 Properties of Complex Numbers

It is important to be able to visualize mathematical concepts and to develop geometric intuition—an ability especially valuable in complex analysis. In this section we define and give a geometric interpretation for several concepts: the *absolute value, argument, polar representation*, and *complex conjugate* of a complex number.

Addition of Complex Numbers In the preceding section a complex number was defined to be a point in the plane $\mathbb{R}^2$. Thus, a complex number may be thought of geometrically as a (two-dimensional) vector and pictured as an arrow from the origin to the point in $\mathbb{R}^2$ given by the complex number (see Figure 1.2.1).

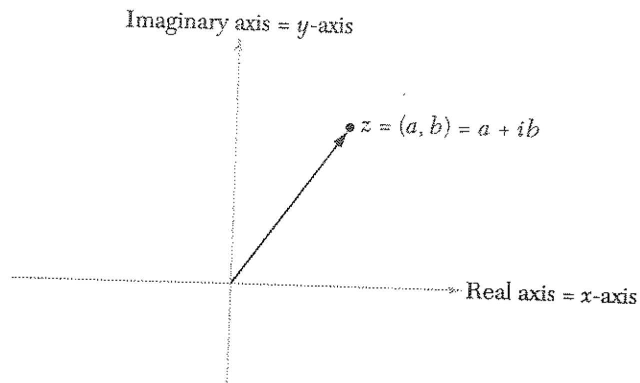

Figure 1.2.1: Vector representation of complex numbers.

Because the points $(x, 0) \in \mathbb{R}^2$ correspond to real numbers, the horizontal or x axis is called the ***real axis***. Similarly, the vertical axis (the y axis) is called the ***imaginary axis***, because points on it have the form $iy = (0, y)$ for y real.

As we already saw in Figure 1.1.1, the addition of complex numbers can be pictured as addition of vectors (an explicit example is given in Figure 1.2.2).

Polar Representation of Complex Numbers To understand the geometric meaning of multiplying two complex numbers, we will write them in what is called polar coordinate form. Recall that the ***length*** of the vector $(a, b) = a + ib$ is defined to be $\sqrt{a^2 + b^2}$. Suppose the vector makes an angle θ with the positive direction of the real axis, where $0 \leq \theta < 2\pi$ (see Figure 1.2.3).

Thus, $\tan\theta = b/a$. Since $a = r\cos\theta$ and $b = r\sin\theta$, we have

$$a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

This way of writing the complex number is called the ***polar coordinate representation***. The length of the vector $z = (a, b) = a + ib$ is denoted $|z|$ and is called the ***norm***, or ***modulus***, or ***absolute value*** of z. The angle θ is called the ***argument*** of the complex number and is denoted $\theta = \arg z$.

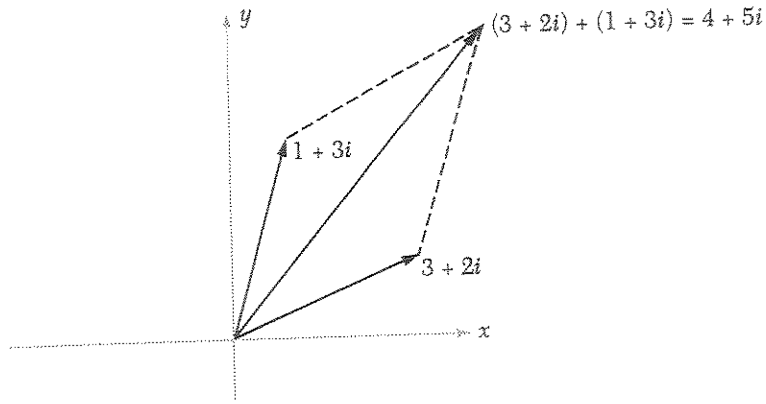

Figure 1.2.2: Addition of complex numbers.

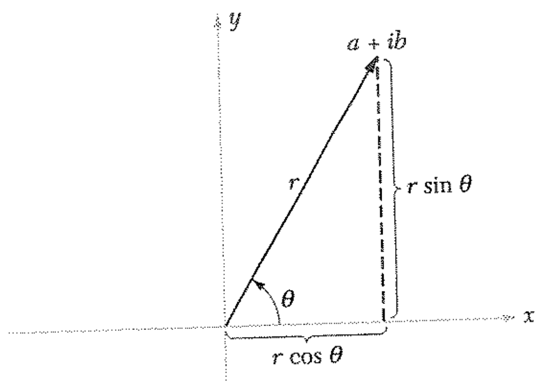

Figure 1.2.3: Polar coordinate representation of complex numbers.

If we restrict θ to the interval $0 \leq \theta < 2\pi$, then each nonzero complex number has an unambiguously defined argument. (We learn this in trigonometry.) However, it is clear that we can add integral multiples of 2π to θ and still obtain the same complex number. In fact, we shall find it convenient to be flexible in our requirements for the values that θ is to assume. For example, we could equally well allow the range of θ to be $-\pi < \theta \leq \pi$. Such an interval must always be specified or be clearly understood.

Once an interval of length 2π is specified, then for each $z \neq 0$, a unique θ is determined that lies within that specified interval. It is clear that any $\theta \in \mathbb{R}$ can be brought into our specified interval by the addition of some (positive or negative) integral multiple of 2π. For these reasons it is sometimes best to think of $\arg z$ as the set of possible values of the angle. If θ is one possible value, then so is $\theta + 2\pi n$ for any integer n, and we can sometimes think of $\arg z$ as $\{\theta + 2\pi n \mid n \text{ is an integer}\}$. Specification of a particular range for the angle is known as choosing a ***branch of the argument***.

Multiplication of Complex Numbers The polar representation of complex numbers helps us understand the geometric meaning of the product of two complex numbers. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2)] + i[(\cos\theta_1 \cdot \sin\theta_2 + \cos\theta_2 \cdot \sin\theta_1)] \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)], \end{aligned}$$

by the addition formula for the sine and cosine functions used in trigonometry. Thus, we have proven

Proposition 1.2.1 *For any complex numbers z_1 and z_2,*

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad \textit{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}.$$

In other words, the product of two complex numbers is the complex number that has a length equal to the product of the lengths of the two complex numbers and an argument equal to the sum of the arguments of those numbers. This is the basic geometric representation of complex multiplication (see Figure 1.2.4).

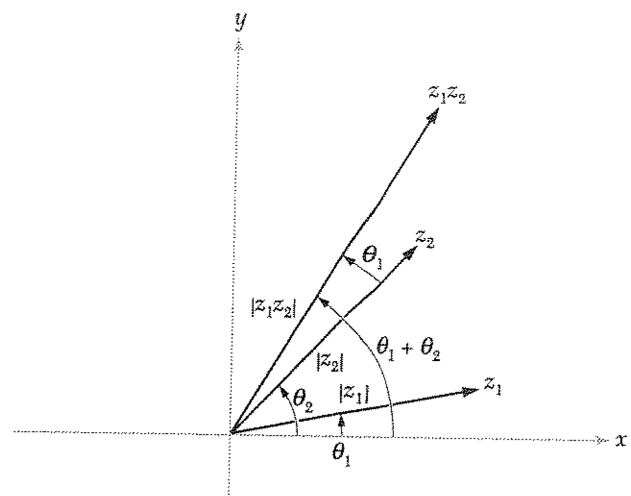

Figure 1.2.4: Multiplication of complex numbers.

The second equality in Proposition 1.2.1 means that the sets of possible values for the left and right sides are the same, that is, that the two sides can be made to agree by the addition of the appropriate multiple of 2π to one side. If a particular branch is desired and $\arg z_1 + \arg z_2$ lies outside the interval that we specify, we should adjust it by a multiple of 2π to bring it within that interval. For example, if our interval is $[0, 2\pi[$ and $z_1 = -1$ and $z_2 = -i$, then $\arg z_1 = \pi$ and $\arg z_2 = 3\pi/2$ (see Figure 1.2.5), but $z_1 z_2 = i$, so $\arg z_1 z_2 = \pi/2$, and $\arg z_1 + \arg z_2 = \pi + 3\pi/2 = 2\pi + \pi/2$. We can obtain the correct answer by subtracting 2π to bring it within the interval $[0, 2\pi[$.

Multiplication of complex numbers can be analyzed in another useful way. Let $z \in \mathbb{C}$ and define $\psi_z : \mathbb{C} \to \mathbb{C}$ by $\psi_z(w) = wz$; that is, ψ_z is the map "multiplication

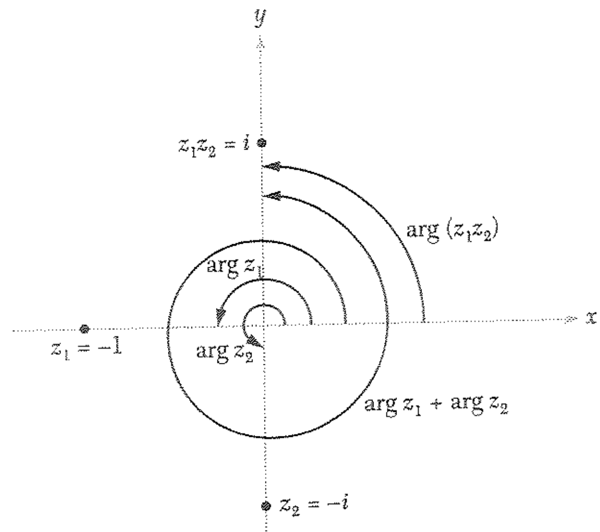

Figure 1.2.5: Multiplication of the complex numbers -1 and $-i$.

by z". By Proposition 1.2.1, *the effect of this map is to rotate a complex number through an angle equal to* $\arg z$ *in the counterclockwise direction and to stretch its length by the factor* $|z|$. For example, ψ_i (multiplication by i) rotates complex numbers by $\pi/2$ in the counterclockwise direction (see Figure 1.2.6).

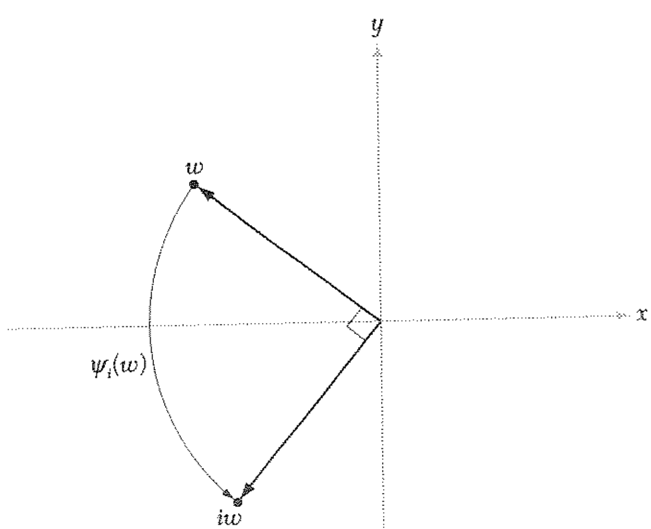

Figure 1.2.6: Multiplication by i.

The map ψ_z is a linear transformation on the plane, in the sense that $\psi_z(\lambda w_1 + \mu w_2) = \lambda\psi_z(w_1) + \mu\psi_z(w_2)$, where λ, μ are real numbers and w_1, w_2 are complex numbers. Any linear transformation of the plane to itself can be represented by a matrix, as we learn in linear algebra. If $z = a + ib = (a, b)$, then the matrix of ψ_z is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}$$

(see Exercise 10, §1.1).

De Moivre's Formula The formula we derived for multiplication, using the polar coordinate representation, provides more than geometric intuition. We can use it to obtain a formula that enables us to find the nth roots of any complex number.

Proposition 1.2.2 (De Moivre's Formula) *If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof By Proposition 1.2.1,

$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] = r^2(\cos 2\theta + i\sin 2\theta).$$

Multiplying again by z gives

$$z^3 = z \cdot z^2 = r \cdot r^2[\cos(2\theta + \theta) + i\sin(2\theta + \theta)] = r^3(\cos 3\theta + i\sin 3\theta).$$

This procedure may be continued by induction to obtain the desired result for any integer n. ■

Let w be a complex number; that is, let $w \in \mathbb{C}$. Using de Moivre's formula will help us solve the equation $z^n = w$ for z when w is given. Suppose that $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\psi + i\sin\psi)$. Then de Moivre's formula gives $z^n = \rho^n(\cos n\psi + i\sin n\psi)$. It follows that $\rho^n = r = |w|$ by uniqueness of the polar representation and $n\psi = \theta + k(2\pi)$, where k is some integer. Thus

$$z = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right) + i\sin\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right)\right].$$

Each value of $k = 0, 1, \ldots, n-1$ gives a different value of z. Any other value of k merely repeats one of the values of z corresponding to $k = 0, 1, 2, \ldots, n-1$. Thus there are exactly n nth roots of a (nonzero) complex number.

An example will help illustrate how to use this theory. Consider the problem of finding the three solutions to the equation $z^3 = 1 = 1(\cos 0 + i\sin 0)$. The preceding formula gives them as follows:

$$z = \cos\frac{k2\pi}{3} + i\sin\frac{k2\pi}{3},$$

where $k = 0, 1, 2$. In other words, the solutions are

$$z = 1, \quad -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

This procedure forfinding roots is summarized as follows.

Corollary 1.2.3 *Let w be a nonzero complex number with polar representation $w = r(\cos\theta + i\sin\theta)$. Then the nth roots of w are given by the n complex numbers*

$$z_k = \sqrt[n]{2}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \qquad k = 0, 1, \ldots, n-1.$$

As a special case of this formula we note that the n roots of 1 (that is, the nth roots of unity) are 1 and $n - 1$ other points equally spaced around the unit circle, as illustrated in Figure 1.2.7 for the case $n = 8$.

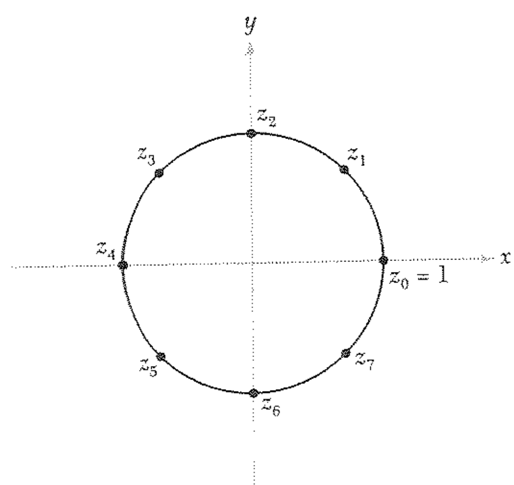

Figure 1.2.7: The eighth roots of unity.

Complex Conjugation Subsequent chapters will include many references to the simple idea of conjugation, which is defined as follows: If $z = a + ib$, then $\bar{z}$, the ***complex conjugate*** of z, is defined by $\bar{z} = a - ib$. Complex conjugation can be pictured geometrically as reflection in the real axis (see Figure 1.2.8).

The next proposition summarizes the main properties of complex conjugation.

Proposition 1.2.4 *The following properties hold for complex numbers:*

(i) $\overline{z + z'} = \bar{z} + \overline{z'}$.

(ii) $\overline{zz'} = \bar{z}\overline{z'}$.

(iii) $\overline{z/z'} = \bar{z}/\overline{z'}$ *for* $z' \neq 0$.

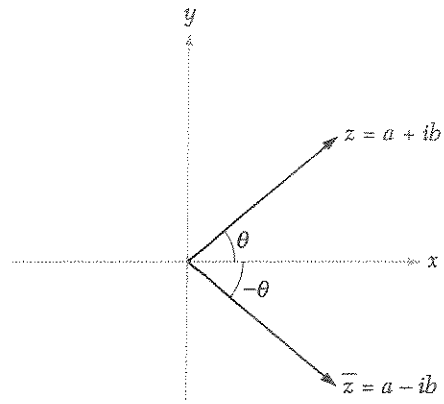

Figure 1.2.8: Complex conjugation.

(iv) $z\bar{z} = |z|^2$ *and hence if* $z \neq 0$, *we have* $z^{-1} = \bar{z}/|z|^2$.

(v) $z = \bar{z}$ *if and only if* z *is real.*

(vi) $\text{Re}\ z = (z + \bar{z})/2$ *and* $\text{Im}\ z = (z - \bar{z})/2i$.

(vii) $\bar{\bar{z}} = z$.

Proof

(i) Let $z = a + ib$ and let $z' = a' + ib'$. Then $z + z' = a + a' + i(b + b')$, and so $\overline{z + z'} = (a + a') - i(b + b') = a - ib + a' - ib' = \bar{z} + \overline{z'}$.

(ii) Let $z = a + ib$ and let $z' = a' + ib'$. Then

$$\overline{zz'} = \overline{(aa' - bb') + i(ab' + a'b)} = (aa' - bb') - i(ab' + a'b).$$

On the other hand, $\bar{z}\overline{z'} = (a - ib)(a' - ib') = (aa' - bb') - i(ab' + a'b)$.

(iii) By (ii) we have $\overline{z'}\,\overline{z/z'} = \overline{z'z/z'} = \bar{z}$. Hence, $\overline{z/z'} = \bar{z}/\overline{z'}$.

(iv) $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$.

(v) If $a + ib = a - ib$, then $ib = -ib$, and so $b = 0$.

(vi) This assertion is clear by the definition of $\bar{z}$.

(vii) This assertion is also clear by the definition of complex conjugation. ■

The absolute value of a complex number $|z| = |a + ib| = \sqrt{a^2 + b^2}$, which is the usual Euclidean length of the vector representing the complex number, has already been defined. From Proposition 1.2.4(iv), note that $|z|$ is also given by $|z|^2 = z\bar{z}$. The absolute value of a complex number is encountered throughout complex analysis; the following properties of the absolute value are quite basic.

Proposition 1.2.5 (i) $|zz'| = |z| \cdot |z'|$.

(ii) *If* $z' \neq 0$, *then* $|z/z'| = |z|/|z'|$.

(iii) $-|z| \leq \operatorname{Re} z \leq |z|$ *and* $-|z| \leq \operatorname{Im} z \leq |z|$; *that is,* $|\operatorname{Re} z| \leq |z|$ *and* $|\operatorname{Im} z| \leq |z|$.

(iv) $|\bar{z}| = |z|$.

(v) $|z + z'| \leq |z| + |z'|$.

(vi) $|z - z'| \geq ||z| - |z'||$.

(vii) $|z_1 w_1 + \ldots + z_n w_n| \leq \sqrt{|z_1|^2 + \ldots + |z_n|^2}\sqrt{|w_1|^2 + \ldots + |w_n|^2}$.

Statement (iv) is clear geometrically from Figure 1.2.8, (v) is called the ***triangle inequality*** for vectors in $\mathbb{R}^2$ (see Figure 1.2.9) and (vii) is referred to as the ***Cauchy-Schwarz inequality***. By repeated application of (v) we get the general statement $|z_1 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|$.

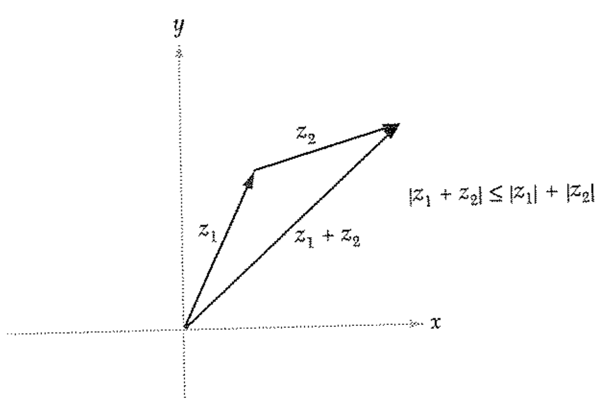

Figure 1.2.9: Triangle inequality.

Proof

(i) This equality was shown in Proposition 1.2.1.

(ii) By (i), $|z'||z/z'| = |z' \cdot (z/z')| = |z|$, so $|z/z'| = |z|/|z'|$.

(iii) If $z = a + ib$, then $-\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2}$ since $b^2 \geq 0$. The other inequality asserted in (iii) is similarly proved.

(iv) If $z = a + ib$, then $\bar{z} = a - ib$, and we clearly have $|z| = \sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = |\bar{z}|$.

(v) By Proposition 1.2.4(iv),

$$\begin{aligned}|z+z'|^2 &= (z+z')\overline{(z+z')}\\ &= (z+z')(\bar{z}+\overline{z'})\\ &= z\bar{z}+z'\overline{z'}+z'\bar{z}+z\overline{z'}.\end{aligned}$$

But $z\overline{z'}$ is the conjugate of $z'\bar{z}$ (Why?), so by Proposition 1.2.4(vi) and (iii) in this proof,

$$|z|^2+|z'|^2+2\operatorname{Re} z'\bar{z} \le |z|^2+|z'|^2+2|z'\bar{z}| = |z|^2+|z'|^2+2|z||z'|.$$

But this equals $(|z|+|z'|)^2$, so we get our result.

(vi) By applying (v) to z' and $z-z'$ we get

$$|z| = |z'+(z-z')| \le |z'|+|z-z'|,$$

so $|z-z'| \ge |z|-|z'|$. By interchanging the roles of z and z', we similarly get $|z-z'| \ge |z'|-|z| = -(|z|-|z'|)$, which is what we originally claimed.

(vii) This inequality is less evident and the proof of it requires a slight mathematical trick (see Exercise 22 for a different proof). Let us suppose that not all the $w_k = 0$ (or else the result is clear). Let

$$v=\sum_{k=1}^{n}|z_k|^2 \quad t=\sum_{k=1}^{n}|w_k|^2 \quad s=\sum_{k=1}^{n}z_k w_k \quad \text{and} \quad c=s/t.$$

Now consider the sum

$$\sum_{k=1}^{n}|z_k - c\bar{w}_k|^2$$

which is ≥ 0 and equals

$$\begin{aligned}v+|c|^2t-c\sum_{k=1}^{n}\bar{z}_k\bar{w}_k-\bar{c}\sum_{k=1}^{n}z_k w_k &= v+|c|^2t-2\operatorname{Re}\bar{c}s\\ &= v+\frac{|s|^2}{t}-2\operatorname{Re}\frac{\bar{s}s}{t}.\end{aligned}$$

Since t is real and $s\bar{s}=|s|^2$ is real, $v+(|s|^2/t)-2(|s|^2/t) = v-|s|^2/t \ge 0$. Hence $|s|^2 \le vt$, which is the desired result. ■

Worked Examples

Example 1.2.6 *Solve* $z^8 = 1$ *for* z.

Solution Since $1 = \cos k2\pi + i \sin k2\pi$ when k equals any integer, Corollary 1.2.3 gives

$$
\begin{aligned}
z &= \cos \frac{k2\pi}{8} + i \sin \frac{k2\pi}{8} \qquad k = 0, 1, 2, \ldots, 7 \\
&= 1, \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, i, \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -1, \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, -i, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.
\end{aligned}
$$

These may be pictured as points evenly spaced on the circle in the complex plane (see Figure 1.2.10).

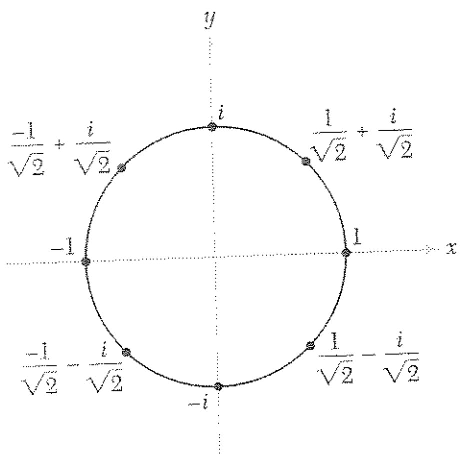

Figure 1.2.10: The eight 8th roots of unity.

Example 1.2.7 *Show that*

$$\overline{\left[\frac{(3+7i)^2}{(8+6i)}\right]} = \frac{(3-7i)^2}{(8-6i)}.$$

Solution The point here is that it is not necessary first to work out $\overline{(3+7i)^2/(8+6i)}$ if we simply use the properties of complex conjugation, namely, $\overline{z^2} = (\bar{z})^2$ and $\overline{z/z'} = \bar{z}/\overline{z'}$. Thus we obtain

$$\overline{\left[\frac{(3+7i)^2}{(8+6i)}\right]} = \frac{\overline{(3+7i)^2}}{\overline{(8+6i)}} = \frac{\left(\overline{3+7i}\right)^2}{\overline{(8+6i)}} = \frac{(3-7i)^2}{(8-6i)}.$$

Example 1.2.8 *Show that the maximum absolute value of z^2+1 on the unit disk $|z| \le 1$ is 2.*

Solution By the triangle inequality, $|z^2+1| \le |z^2| + 1 = |z|^2 + 1 \le 1^2 + 1 = 2$, since $|z| \le 1$ thus $|z^2+1|$ does not exceed 2 on the disk. Since the value 2 is achieved at $z = 1$, the maximum is 2.

Example 1.2.9 *Express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ using de Moivre's formula.*

Solution De Moivre's formula for $r = 1$ and $n = 3$ gives the identity

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

The left side of this equation, when expanded (see Exercise 14 of §1.1), becomes

$$\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

By equating real and imaginary parts, we get

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and the additional formula

$$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta.$$

Example 1.2.10 *Write the equation of a straight line, of a circle, and of an ellipse using complex notation.*

Solution The straight line is most conveniently expressed in parametric form: $z = a + bt, a, b \in \mathbb{C}, t \in \mathbb{R}$, which represents a line in the direction of b and passing through the point a.

The circle can be expressed as $|z - a| = r$ (radius r, center a).

The ellipse can be expressed as $|z - d| + |z + d| = 2a$; the foci are located at $\pm d$ and the semimajor axis equals a.

These equations, in which $|\cdot|$ is interpreted as length, coincide with the geometric definitions of these loci.

Exercises

1. Solve the following equations:

   (a) $z^5 - 2 = 0$

   (b) $z^4 + i = 0$

2. Solve the following equations:

   (a) $z^6 + 8 = 0$

   (b) $z^3 - 4 = 0$

3. What is the complex conjugate of $(3 + 8i)^4/(1 + i)^{10}$?

4. What is the complex conjugate of $(8 - 2i)^{10}/(4 + 6i)^5$?

5. Express $\cos 5x$ and $\sin 5x$ in terms of $\cos x$ and $\sin x$.

6. Express $\cos 6x$ and $\sin 6x$ in terms of $\cos x$ and $\sin x$.

7. Find the absolute value of $[i(2+3i)(5-2i)]/(-2-i)$.

8. Find the absolute value of $(2-3i)^2/(8+6i)^2$.

9.• Let w be an nth root of unity, $w \neq 1$. Show that $1+w+w^2+\ldots+w^{n-1}=0$.

10. Show that the roots of a polynomial with real coefficients occur in conjugate pairs.

11. If $a, b \in \mathbb{C}$, prove the ***parallelogram identity***: $|a-b|^2+|a+b|^2 = 2(|a|^2+|b|^2)$.

12. Interpret the identity in Exercise 11 geometrically.

13. When does equality hold in the triangle inequality $|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n|$? Interpret your result geometrically.

14. Assuming either $|z|=1$ or $|w|=1$ and $\bar{z}w \neq 1$, prove that

$$\left|\frac{z-w}{1-\bar{z}w}\right| = 1.$$

15. Does $z^2 = |z|^2$? If so, prove this equality. If not, for what z is it true?

16.• Letting $z = x+iy$, prove that $|x|+|y| \leq \sqrt{2}|z|$.

17.• Let $z = a+ib$ and $z' = a'+ib'$. Prove that $|zz'| = |z||z'|$ by evaluating each side.

18. Prove the following:

(a) $\arg \bar{z} = -\arg z \pmod{2\pi}$

(b) $\arg(z/w) = \arg z - \arg w \pmod{2\pi}$

(c) $|z| = 0$ if and only if $z = 0$

19. What is the equation of the circle with radius 3 and center $8+5i$ in complex notation?

20. Using the formula $z^{-1} = \bar{z}/|z|^2$, show how to construct z^{-1} geometrically.

21. Describe the set of all z such that $\operatorname{Im}(z+5) = 0$.

22.• Prove ***Lagrange's identity***:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n}|z_k|^2\right)\left(\sum_{k=1}^{n}|w_k|^2\right) - \sum_{k<j}|z_k\bar{w}_j - z_j\bar{w}_k|^2.$$

Deduce the Cauchy-Schwarz inequality from your proof.

23.* Given $a \in \mathbb{C}$, find the maximum of $|z^n + a|$ for those z with $|z| \leq 1$.

24. Compute the least upper bound (that is, supremum) of the set of all real numbers of the form Re $(iz^3 + 1)$ such that $|z| < 2$.

25.* Prove ***Lagrange's trigonometric identity***:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

(Assume that $\sin(\theta/2) \neq 0$.)

26. Suppose that the complex numbers z_1, z_2, z_3 satisfy the equation

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}.$$

Prove that $|z_2 - z_1| = |z_3 - z_1| = |z_2 - z_3|$. *Hint*: Argue geometrically, interpreting the meaning of each statement.

27. Give a necessary and sufficient condition for

(a) z_1, z_2, z_3 to lie on a straight line.

(b) z_1, z_2, z_3, z_4 to lie on a straight line or a circle.

28. Prove the identity

$$\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\ldots\left(\sin\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Hint: The given product can be written as $1/2^{n-1}$ times the product of the nonzero roots of the polynomial $(1-z)^n - 1$.

29. Let w be an nth root of unity, $w \neq 1$. Evaluate $1 + 2w + 3w^2 + \ldots + nw^{n-1}$.

30. Show that the correspondence of the complex number $z = a + bi$ with the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \psi_z$ noted in the text preceding Proposition 1.2.2 has the following properties:

(a) $\psi_{zw} = \psi_z \psi_w$.

(b) $\psi_{z+w} = \psi_z + \psi_w$.

(c) $\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $\lambda\psi_z = \psi_{\lambda z}$ if λ is real.

(e) $\psi_{\bar{z}} = (\psi_z)^t$ (the transposed matrix).

(f) $\psi_{1/z} = (\psi_z)^{-1}$.

(g) z is real if and only if $\psi_z = (\psi_z)^t$.

(h) $|z| = 1$ if and only if ψ_z is an orthogonal matrix.