Math 4200
Fri 9/9

- go slowly and carefully through the chain rules on page 4 Wed. There are two of them, one for composition of analytic functions, one for an analytic function of a parametric curve.

$$\frac{d}{dz}\, g(f(z)) = g'(f(z))\cdot g'(z)$$

$$\frac{d}{dt}\, f(\gamma(t)) = \underbrace{f'(\gamma(t))}_{\substack{\text{complex}\\ \text{derivative}}}\ \underbrace{\gamma'(t)}_{\text{tangent vector.}}$$

HW for Fri 9/16

§1.5 6b (draw the 1-square diagram to illustrate the differential map), 25, 26, 27, 28

§1.6 1c, 2abc, 3a, 4, 6, 10, 14

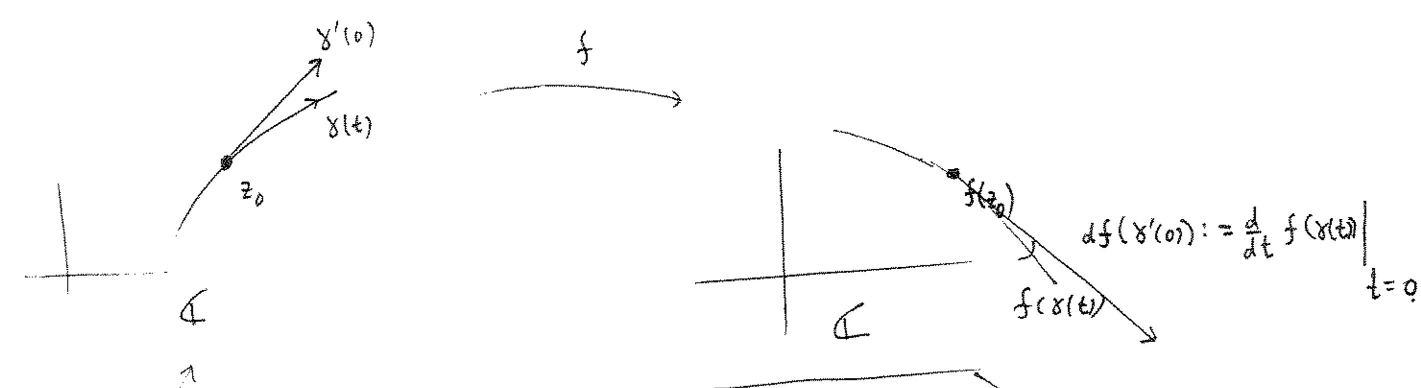
Class exercise 1 in today's notes

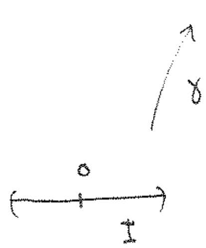
the chain rule for curves explains the infinitesimal behavior of analytic transformations...

one way to discuss this is in terms of the differential map df

$$df_{z_0} : \underbrace{T\mathbb{C}_{z_0}}_{\substack{\text{tangent space to } \mathbb{C} \text{ at } z_0 \\ = \{\gamma'(0) \text{ s.t. } \gamma: I \to \mathbb{C},\ \gamma(0) = z_0\}}} \longrightarrow T\mathbb{C}_{f(z_0)}$$

$$df(\gamma'(0)) := \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$$





The chain rule for curves implies $df(\gamma'(0)) = \underbrace{f'(z_0)}_{re^{i\theta}}\ \underbrace{\gamma'(0)}_{\rho e^{i\phi}}$

thus the differential transformation is a rotation by θ and dilation by r.

Example

$f(z) = z^2 \qquad f'(z) = 2z$

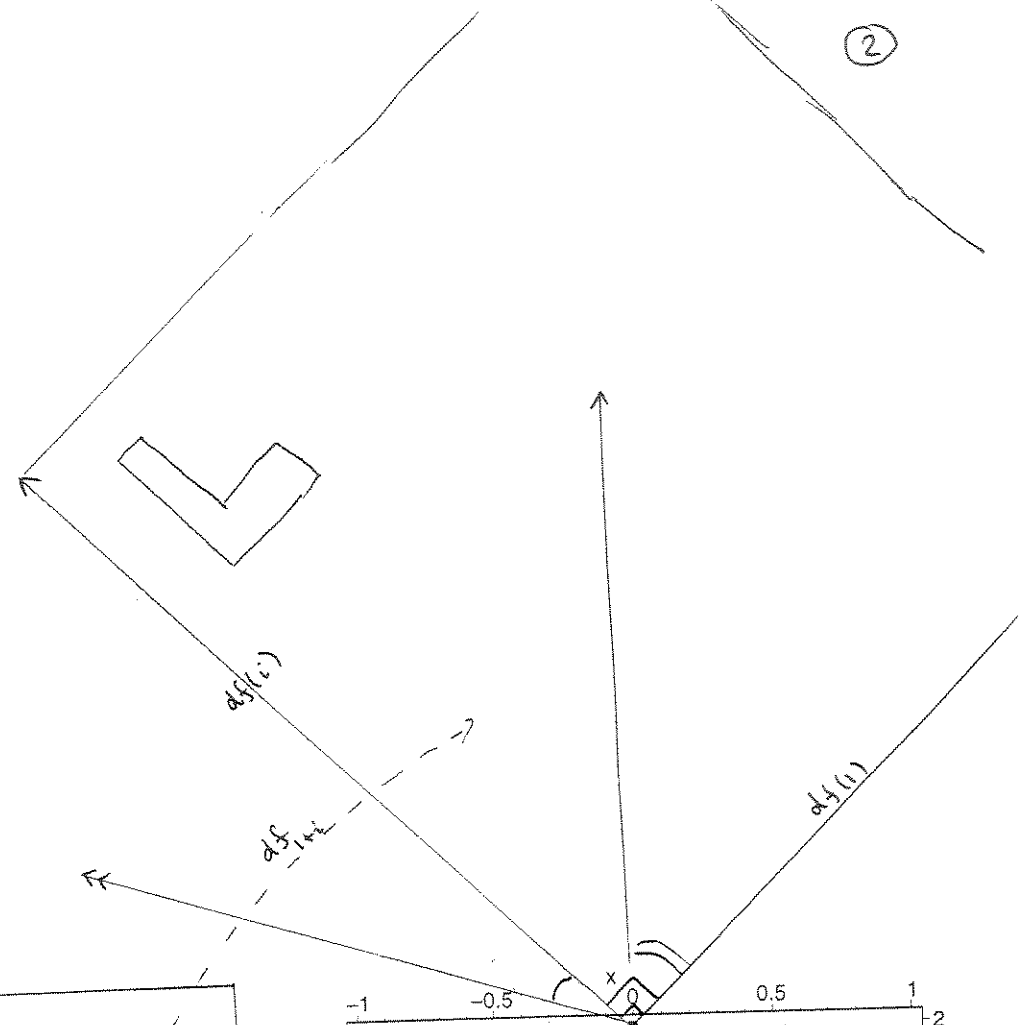
$z_0 = 1+i \;,\; f(z_0) = (1+i)^2 = 2i$

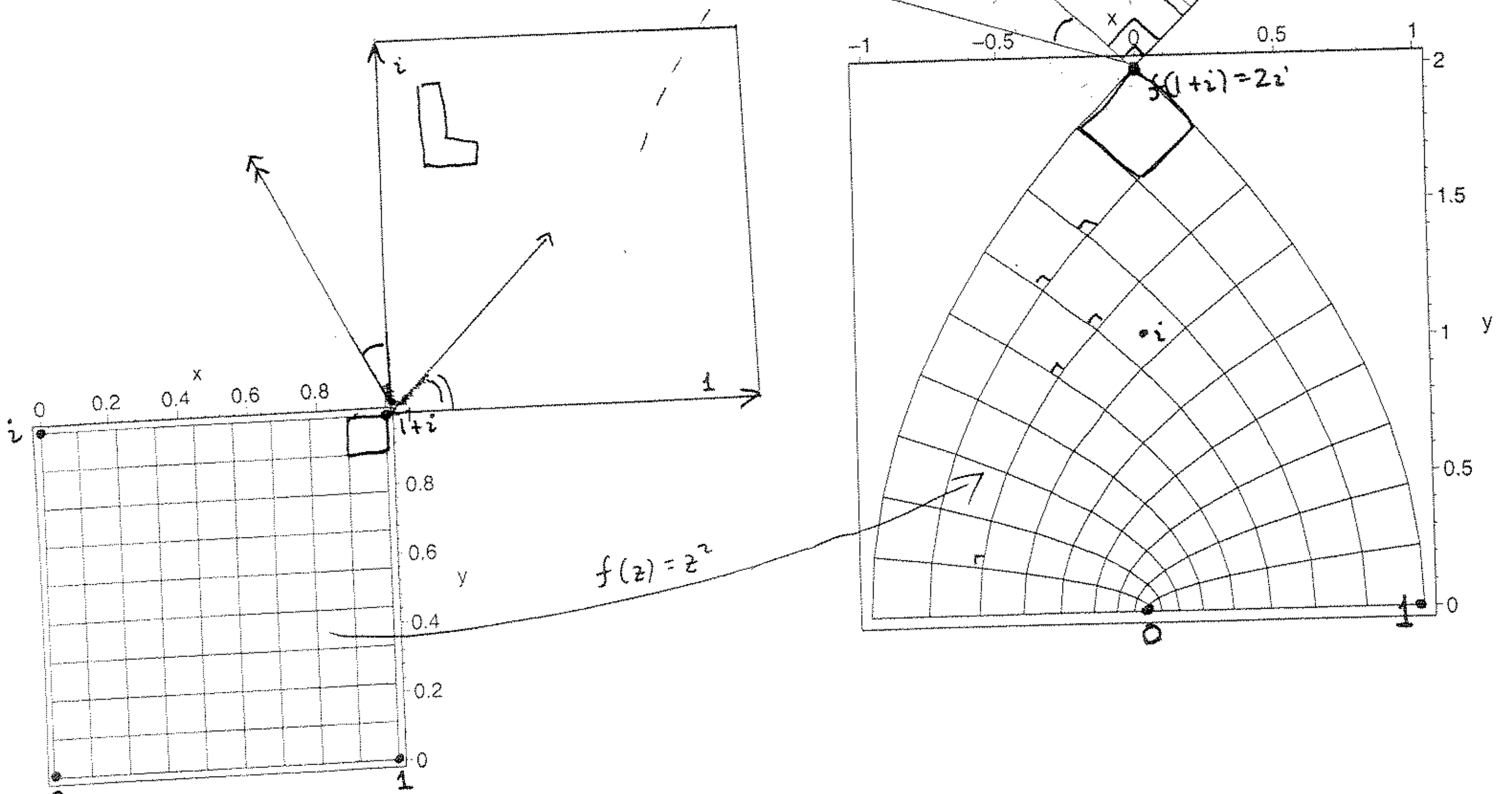
If $\gamma(0) = 1+i$

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = f'(1+i)\,\gamma'(0)$$

so $df_{1+i}(\gamma'(0)) = 2(1+i)\gamma'(0)$
$= 2\sqrt{2}\, e^{i\pi/4}\, \gamma'(0)$

$\boxed{\begin{array}{l}\text{so } df_{1+i} \text{ is rotation dilation,} \\ \text{with rot} = \pi/4 \\ \text{stretch} = 2\sqrt{2}\end{array}}$





Analytic maps are called <u>conformal</u> wherever $f'(z_0) \neq 0$ because the differential map $df : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$ is a rotation-dilation, so "<u>preserves shape</u>".

(on small scales, image shapes are very close to scaled rotations of domain shapes, because $f(z_0+h) = f(z_0) + df_{z_0} \cdot h + \varepsilon(h)h$)

In particular, angles between domain tangent vectors and image tangent vectors are equal. also, the length scaling factor $|f'(z_0)|$ is the same for all tangent vectors at z_0, and their image-tangent vectors under the differential map. ("uniform scaling").

<u>Class exercise 1</u>

Under the correspondence $f(x+iy) = u(x,y) + i\,v(x,y)$

$$\tilde{F}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}.$$

there is the analogous differential map

$$dF : T_{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}\mathbb{R}^2 \to T_{\begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix}}\mathbb{R}^2$$

$$T(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad T(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$dF(T'(0)) := \frac{d}{dt} F(T(t))\Big|_{t=0}$$

$$= \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} x'(0) \\ y'(0) \end{bmatrix} \quad \text{(Real Chain rule)}$$

$$= \underbrace{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}_{\text{rotation dilation.}}\begin{bmatrix} x'(0) \\ y'(0) \end{bmatrix}$$

<u>If</u> f is analytic at z_0, with $f'(z_0) = a+bi$.

Prove that if dF preserves angles and orientation, then dF is actually a rotation-dilation, so you automatically get uniform scaling, <u>and</u> f analytic at z_0. (Sometimes people call conformal maps "angle preserving".)

Precisely:

$$\measuredangle\, dF(\vec{v}, \vec{w}) = \measuredangle\, \vec{v}, \vec{w} \quad \forall\, \vec{v}, \vec{w} \in T_{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}\mathbb{R}^2$$

and $\det[dF] > 0$

$\Rightarrow$ dF is a rotation-dilation

hint: use the dot product and good choices for $\vec{v}, \vec{w}$ to deduce the matrix for dF.

Now do <u>page 5 Wed</u>