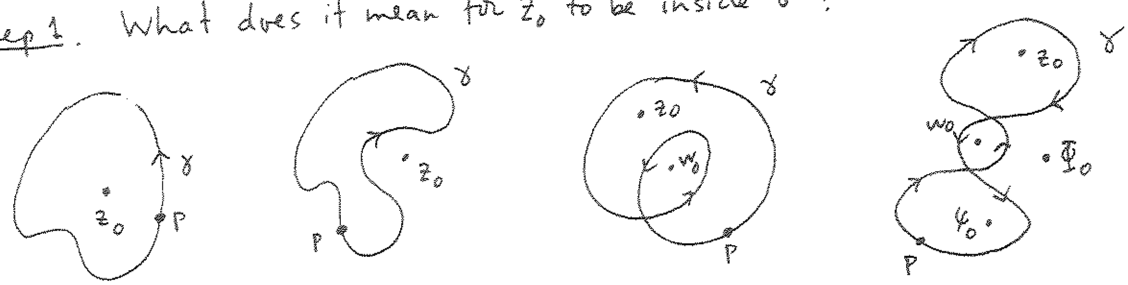
Math 4200
Fri. 9/30

Bring 2007 midterm exam to class on Monday, as we review key concepts for Wed. exam

§2.4 (not on Wed. exam).

This section is about the magic fact that if a piecewise C^1 closed contour γ is given, if $f(z)$ is analytic in an open simply-connected domain A containing γ, and if z_0 is inside γ, then $f(z_0)$ can be computed with an appropriate contour integral around γ.
This is the Cauchy Integral Formula and is the basis for many amazing facts about analytic functions, and corollaries important in diverse pure/applied applications

Step 1. What does it mean for z_0 to be inside γ?



Def If γ is a continuous closed path in $\mathbb{C}$, $\gamma : [a,b] \to \mathbb{C}$ cont, $\gamma(a) = \gamma(b) = P$

and if $z_0 \notin \text{image}(\gamma)$
Then the winding number of γ about z_0, also called the index of γ relative to z_0 and written $I(\gamma; z_0)$ is how many times γ winds about z_0 in the counterclockwise direction.

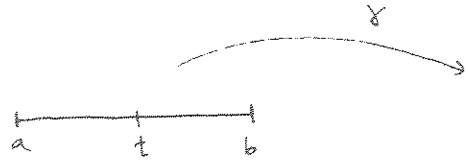
This number is usually easy to compute if you can see the image curve γ.

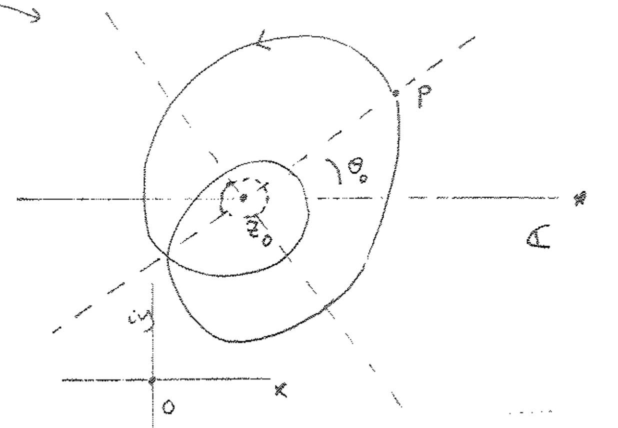
Exercise 1: Compute the winding numbers in the examples above.

Def z_0 is inside γ iff $I(\gamma; z_0) \neq 0$.

Lemma $I(\gamma; z_0)$ is well defined for any continuous closed curve γ with $z_0 \notin \text{image}(\gamma)$.

proof:





γ is uniformly continuous and $z_0 \notin \text{image}(\gamma)$ implies γ stays a uniform distance away from z_0

Pick a choice θ_0 for the polar coordinate angle for $\gamma(a) - z_0$, i.e. for $\arg(\gamma(a) - z_0)$ (determined up to a multiple of 2π).

- We claim there is a unique way to continue $\theta \overset{=\Theta(t)}{}$ continuously so that $\gamma(t) - z_0 = |\gamma(t) - z_0| e^{i\theta(t)}$. $\forall\, t \in [a,b]$.

- And then, $I(\gamma; z_0) := \frac{1}{2\pi}(\Theta(b) - \Theta(a))$ is well defined for any such $\Theta(t)$.

First bullet point: $\exists$: a continuous extension $\Theta(t)$:

consider the open half plane $\{ z \mid \theta_0 - \pi/2 < \arg(z - z_0) < \theta_0 + \pi/2 \}$ as indicated above.

As long as $\gamma(t)$ lies in this half plane,

$\Theta(t) := \arg(\gamma(t) - z_0)$ is continuous.

Let t_1 be the 1st $t > a$ with

$$\underbrace{\theta_0 - \tfrac{\pi}{2} = \arg(z - z_0)}_{\gamma(t_1)} \quad \text{or} \quad \underbrace{\theta_0 + \tfrac{\pi}{2} = \arg(z - z_0)}_{\gamma(t_1)}$$

(if such t_1 exists)

Then extend $\Theta(t)$ for $t > t_1$ using the neighbor halfplane

$$\arg(\gamma(t_1) - z_0) - \pi/2 < \arg(z - z_0) < \arg(\gamma(t_1) - z_0) + \pi/2.$$

Continue inductively to $t_2, t_3, \dots$ if necessary.
Because $\gamma(t)$ is uniformly continuous and because $\exists\, r > 0$ s.t. $|\gamma(t) - z_0| \geq r$, this process terminates after a finite number of steps, with $\Theta(t) = \arg(\gamma(t) - z_0)$ defined and continuous for $a \leq t \leq b$, so that $\gamma(t) - z_0 = |\gamma(t) - z_0| e^{i\Theta(t)}$.

<u>uniqueness</u>: If $\theta_1(t)$, $\theta_2(t)$ are any two continuous functions on $[a,b]$ so that $\arg(\gamma(t)-z_0) = \theta_1(t), \theta_2(t)$ $\forall\ a \le t \le b$

then $\theta_2(t) - \theta_1(t)$ must be a continuous function which is also always an integer multiple of 2π.

Thus $\theta_2(t) - \theta_1(t) = 2\pi k$ for $k \in \mathbb{Z}$ fixed.

second bullet pt: This proves <u>uniqueness</u>, and that $I(\gamma; z_0)$ is <u>well-defined</u> regardless of initial choice of θ_0, since

$$\theta_2(b) - \theta_2(a) = \Phi_1(b) - \theta_1(a).$$

<u>Theorem</u> If γ, z_0 are as above, and if γ is also piecewise C^1, then index can be computed with a contour integral:

$$\boxed{I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz}$$

motivation: locally, in polar coords $z = z_0 + re^{i\theta}$ $\quad \left(\begin{matrix} z = \gamma(t) \\ r = r(t),\ \theta = \theta(t) \end{matrix}\right)$

$$dz = (dr)e^{i\theta} + re^{i\theta}\,i\,d\theta$$

$$\Rightarrow \frac{dz}{z - z_0} = \frac{dr\, e^{i\theta} + re^{i\theta}\,i\,d\theta}{re^{i\theta}}$$

$$\frac{dz}{z - z_0} = \frac{dr}{r} + i\,d\theta$$

proof of theorem: let $a \le s \le b$

let θ_0 and $\theta(t)$ as above. If γ is C^1, then for $a \le s \le t_1$,

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\gamma'(t)\,dt \underset{\uparrow}{=} \frac{1}{2\pi i}\int_a^s \frac{r'(t)e^{i\theta(t)} + r(t)\,i\,e^{i\theta(t)}\theta'(t)}{r(t)e^{i\theta(t)}}\,dt$$

use polar coords in 1st half plane

$\gamma(t) = r(t)\,e^{i\theta(t)}$

$$= \frac{1}{2\pi i}\left[\ln(r(t))\Big]_{t=a}^{s} + i(\theta(s) - \theta(a))\right].$$

continue for $t_1 \le s \le t_2, \ldots$

(and using subintervals also if γ is only piecewise C^1)

Deduce that $\forall\ a \le s \le b$,

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\gamma'(t)\,dt = \frac{1}{2\pi i}\left[\ln\left(\frac{r(s)}{r(a)}\right) + i(\theta(s) - \theta(a))\right]$$

$$\text{for } s = b \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz = \frac{1}{2\pi i}\left[\overset{0}{\cancel{\ln(1)}} + i(\theta(b) - \theta(a))\right]$$

$$= I(\gamma; z_0) \quad \blacksquare$$

Exercise

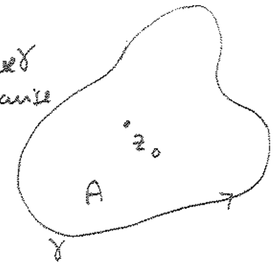
a) Show that for $\gamma(t) = z_0 + re^{int}$ $\quad 0 \le t \le 2\pi$, $n \in \mathbb{Z}$, $n \ne 0$,

(which has $I(\gamma; z_0) = n$),

the contour integral formula $I(\gamma; z_0) = \int_\gamma \frac{dz}{z - z_0}$ agrees.

b) Let $z_0 \in A$, with ∂A p.w. C^1 curve γ counterclockwise

$\uparrow$ open connected.



Show $I(\gamma; z_0) = 1$

Via Green's Theorem.

The amazing

Cauchy Integral formula Let $A \subset \mathbb{C}$ open and simply connected.
$f: A \to \mathbb{C}$ analytic
γ a p.w. C^1 closed contour in A, $z_0 \notin \text{image}(\gamma)$.

Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0} = f(z_0)\, I(\gamma; z_0)$$

so if z_0 is inside γ ($I(\gamma; z_0) \neq 0$),
$f(z_0)$ is determined and computable
from the values of f along γ!

proof: Let
$$g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

g is analytic in $A \setminus \{z_0\}$ and continuous at z_0
So (modified) rectangle lemma holds.
So local antiderivative theorem holds.
So homotopy lemma holds
So contour integrals for $g(z)$ are path independent
So
$$\int_\gamma g(z)\,dz = 0$$

But this $= \displaystyle\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz$ since $z_0 \notin \text{image}(\gamma)$

$$0 = \int_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0)\underbrace{\int_\gamma \frac{dz}{z-z_0}}_{2\pi i\, I(\gamma; z_0)}$$

∎

Remark: if $\gamma = \partial D$ with D closure $\subset A$
and $z_0 \in D$, you can prove the special case of this
$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-z_0}\,dz$$
with Green's Thm and a limiting argument,
assuming also $f \in C^1$