Math 4200
Wed 9/28

- Summarize antiderivative thm in simply-connected domains & deformation thm.

  Go over proof of Homotopy/Homology Lemma

We'll use Monday's notes.
We'll check the analysis claims carefully (these were ones we didn't check carefully back in Chapter 1.

Example let $f(z) = \dfrac{z+8}{z^2-4z}$

let $\gamma(t) = (2+\cos t)e^{2it} \quad 0 \le t \le 2\pi$

Find $\int_\gamma f(z)\,dz$

a) sketch
b) partial fracs
c) use deformation thm for one term, antidiff thm in simply connected domain for the other.

logarithms done carefully

- Let A be open and simply connected in $\mathbb{C}$, with $0 \notin A$. Then $\exists$ well defined branch of $\log z$ analytic in A.

proof: $\frac{1}{z}$ is analytic in A

so $\exists$ antiderivative $G(z)$ s.t. $G'(z) = \frac{1}{z}$ $\forall z \in A$

pick $z_0 \in A$ and adjust G with an additive constant so that

$$e^{G(z_0)} = z_0.$$

We wish to show $e^{G(z)} = z \quad \forall z \in A$.

i.e. $1 = z e^{-G(z)}$

but $D_z\left(z e^{-G(z)}\right) = e^{-G(z)} + z e^{-G(z)}\left(-\frac{1}{z}\right) \equiv 0$

so $z e^{-G(z)}$ is a constant, and this constant is 1 since $z_0 e^{-G(z_0)} = 1$ $\blacksquare$

turns out to follow from f analytic in A, see §2.4

- Let A be open and simply connected in $\mathbb{C}$

  $f: A \to \mathbb{C}$ analytic with $0 \notin \text{range}(f)$, and f' analytic in A

  Then $\exists$ well defined branch of $\log f(z)$ analytic in A, i.e. a function $G(z)$ s.t. $e^{G(z)} = f(z)$ $\forall z \in A$.

proof: $\frac{f'}{f}$ is analytic in A, so $\exists$ antiderivative $G(z)$.

Let $z_0 \in A$. adjust C, s.t. $e^{G(z_0)} = f(z_0)$

Consider $e^{-G(z)} f(z)$, which $= 1$ @ z_0

$$D_z\left(e^{-G(z)} f(z)\right) = e^{-G(z)} f'(z) + e^{-G(z)}\left(-\frac{f'}{f}(z)\right) f(z) \equiv 0 \quad \blacksquare$$

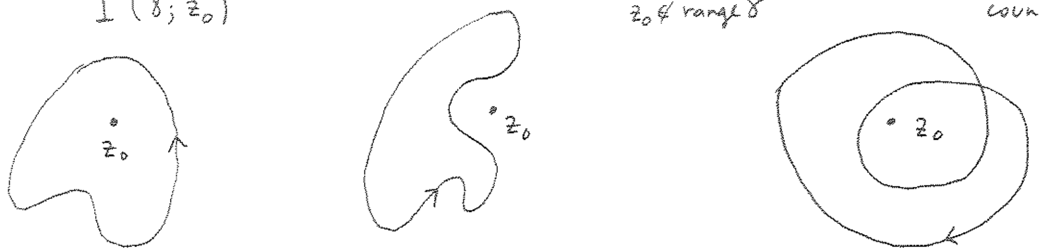
(we probably won't get past here on Wed.)

§2.4 part 1 : <u>index</u> (or <u>winding number</u>) of a closed path γ w.r.t. $z_0 \in A$

Def

- <u>Index</u> (or a <u>winding number</u>) of a closed path γ (piecewise C^1) wrt z_0 ($z_0 \notin$ range γ) : how many times does γ wind around z_0 in counterclockwise direction?

$I(\gamma; z_0)$



$I(\gamma; z_0) = 1$ $\qquad$ $I(\gamma; z_0) = 0$ $\qquad$ $I(\gamma; z_0) = -2$

<u>Def</u>: $I(\gamma; z_0) := \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{1}{z - z_0}\, dz$

- $I(\gamma; z_0)$ is a homotopy invariant in $\mathbb{C} \setminus \{z_0\}$ wrt closed curves, by the deformation theorem.

- If $\gamma^n(t) = z_0 + e^{i(2\pi n t)}$ $\quad 0 \le t\ 1$ $\quad, n \in \mathbb{Z}$

  then $\gamma'(t) = 2\pi n i\, e^{i(2\pi n t)}$

  $z - z_0 = e^{i(2\pi n t)}$

  so, $\dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{1}{z - z_0}\, dz = \dfrac{1}{2\pi i} \int_0^1 2\pi n i\, dt = n$ ✓

(note, we could use $\gamma_r(t) = z_0 + r e^{i(2\pi n t)}$, $r > 0$, as well.
easy to see directly, and by deformation thm.)

<u>Theorem</u> Let γ be a p.w. C^1 closed path in $\mathbb{C}$, $z_0 \notin$ Image(γ).

A) $I(\gamma; z_0) \in \mathbb{Z}$ $\forall$ closed paths γ

B) If $I(\gamma; z_0) = n \in \mathbb{Z}$, then γ is homotopic as a closed path to $\gamma^n(t)$ above

(A follows from B by homotopy theorem, but A is easier to prove.)

proof of A:

let $\gamma : [a,b] \to \mathbb{C}$ piecewise C^1

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds \qquad \Big(\int_\gamma \frac{1}{z - z_0} dz \text{ up to } t \Big)$$

the part of

on the C^1 parts of γ,

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

So

$$\frac{d}{dt}\, e^{-g(t)}(\gamma(t) - z_0) = e^{-g(t)}\Big(\frac{-\gamma'}{\cancel{\gamma - z_0}}\Big)(\cancel{\gamma - z_0}) + e^{-g(t)}\gamma'(t) \equiv 0.$$

so $e^{-g(t)}(\gamma(t) - z_0)$ is piecewise constant and continuous

$\Rightarrow$ constant on $[a,b]$

so $$\underbrace{e^{-g(a)}}_{1}\,\cancel{(\gamma(a) - z_0)} = e^{-g(b)}\,\cancel{(\gamma(b) - z_0)}$$

$\gamma(a) = \gamma(b)$

$\Rightarrow e^{-g(b)} = 1$

$\Rightarrow g(b) = 2\pi i n \;,\; n \in \mathbb{Z}.$

but $I(\gamma; z_0) = \frac{1}{2\pi i} g(b) = n$ ∎

you can use these ideas to actually construct the homotopy in B!