Math 4200
Mon. 9/26
§2.3 cont'd.

On Friday we proved

<u>Rectangle Lemma</u> Let $R = \{x+iy \text{ s.t. } a \le x \le b,\ c \le y \le d\} \subset A$ open.

Let $f : A \to \mathbb{C}$ analytic.
Let $\gamma = \partial R$ (counterclockwise orientation)
Then $\oint_\gamma f(z)\,dz = 0$

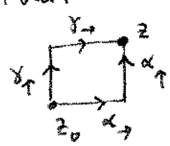
We used Goursat's subdivision argument.
If we had known $f \in C^1(A)$ we could have used Green's Theorem.

Using the Rectangle Lemma we used special coordinate-direction paths to prove

<u>Local antiderivative theorem</u>:

Let $f : D(z_0; r) \to \mathbb{C}$ analytic
Then $\exists\, F : D(z_0; r) \to \mathbb{C}$ s.t. $F'(z) = f(z)\ \forall z \in D(z_0; r)$.

Recall that



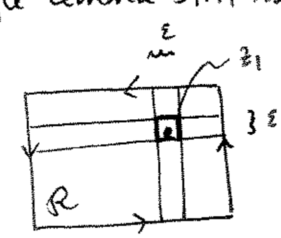
$$F(z) := \int_{\alpha_\to + \alpha_\uparrow} f(\zeta)\,d\zeta = \int_{\gamma_\uparrow + \gamma_\to} f(\zeta)\,d\zeta \qquad \text{(by rect. lemma).}$$

For later: <u>Local antiderivative theorem</u> also holds if $\exists\, z_1 \in D(z_0; r)$ where f is continuous but not known to be analytic, and f analytic on $D(z_0; r) \setminus \{z_1\}$.

proof: Rectangle lemma + f continuous allows construction of F.
Rectangle lemma still holds with these weaker hypotheses: Let $\varepsilon > 0$.

$z_1 \notin R$, no issue

$z_1 \in R$ interior:

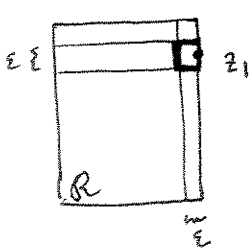


$$\int_{\partial R} f(z)\,dz = \sum_{\text{subrects}} \int_{\partial R_j} f(z)\,dz$$

$$= \int_{\partial R^{z_1}} f(z)\,dz \qquad \text{(because rect. lemma applies in the other subrects)}$$

rectangle containing z_1 → ∂R^{z_1}

$z_1 \in \partial R$:



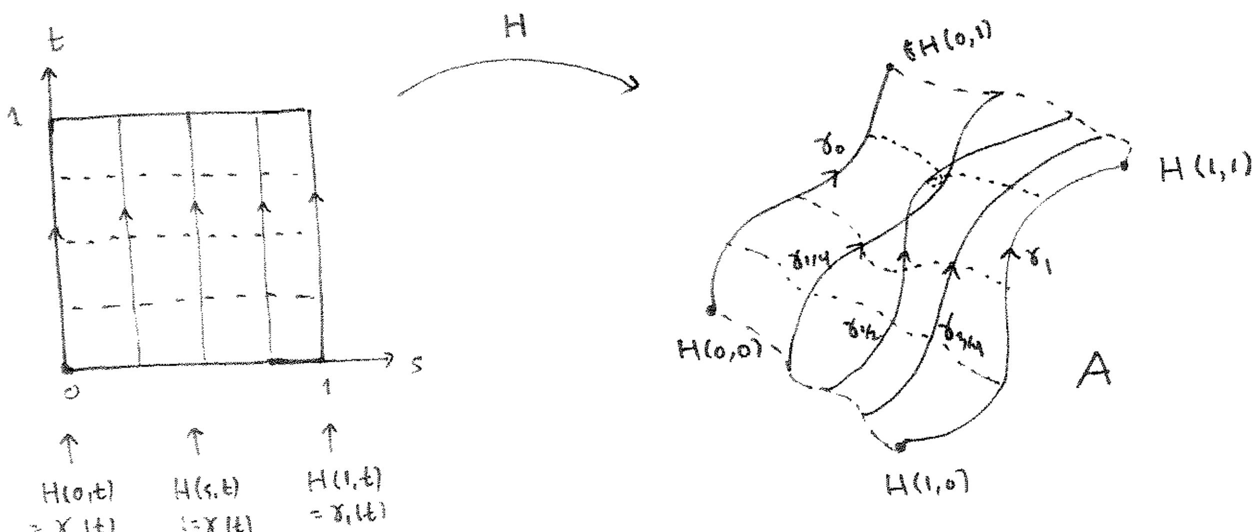
$$\left| \int_{\partial R^{z_1}} f(z)\,dz \right| \le \int_{\partial R^{z_1}} |f(z)|\,|dz| \le \underset{\substack{\uparrow \\ \text{bound on } |f|}}{M}\ \underset{\substack{\uparrow \\ \text{perimeter}}}{(4\varepsilon)} \to 0 \text{ as } \varepsilon \to 0 \qquad \blacksquare$$

Today (& next lecture) we apply local antiderivative theorem,
define homotopy and simply-connected,
and prove (rigorously) a global antiderivative theorem for simply connected domains.
We also prove a deformation theorem about when contour integrals for an analytic f remain unchanged in value with respect to change in contour γ.
This theorem will complement the one we already proved for C^1 analytic functions, using Green's Theorem.

Two curves are homotopic if one of them can be continuously deformed into the other one.

Precisely:

Definition Let $A \subset \mathbb{C}$ open, connected.
Let $\gamma_0, \gamma_1 : [0,1] \to A$ continuous paths.
Then γ_0 is homotopic to γ_1 in A
iff
$\exists$ homotopy $H : \{(s,t) \mid 0 \le s \le 1, 0 \le t \le 1\} \to A$ continuous
s.t. $H(0,t) = \gamma_0(t)$ $\quad 0 \le t \le 1.$
$H(1,t) = \gamma_1(t)$

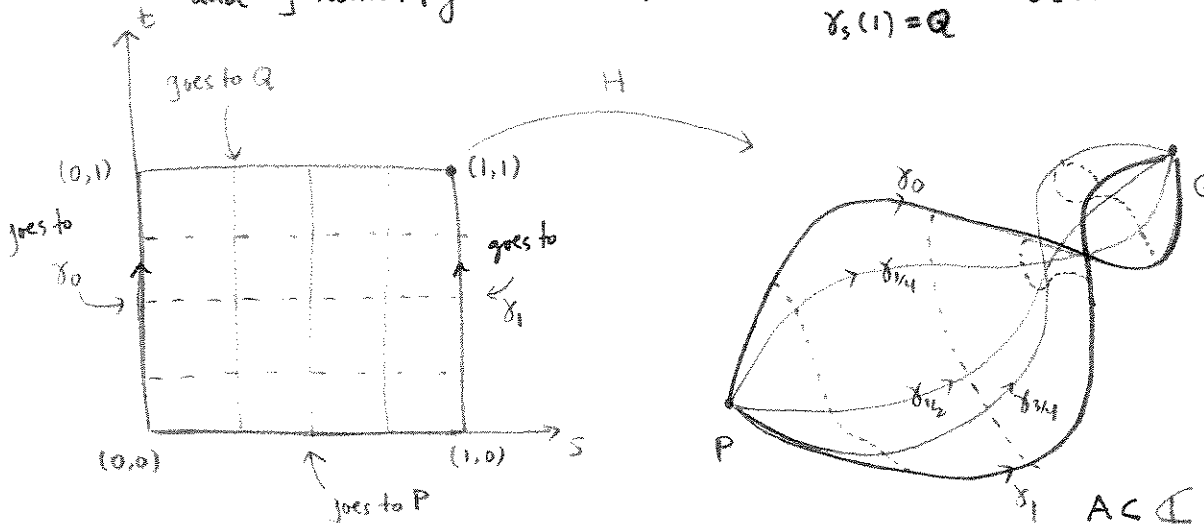


Special cases of homotopic curves:

<u>Def</u> γ_0, γ_1 are <u>homotopic with fixed endpoints in A</u> if

$$\gamma_0(0) = \gamma_1(0) = P$$
$$\gamma_0(1) = \gamma_1(1) = Q$$

and $\exists$ homotopy $H(s,t) = \gamma_s(t)$ s.t. $\gamma_s(0) = P$, $\gamma_s(1) = Q$ $\quad 0 \le s \le 1$



<u>Def</u> Let γ_0, γ_1 as above, except that they are closed curves, $\gamma_0(0) = \gamma_0(1) = P$, $\gamma_1(0) = \gamma_1(1) = Q$

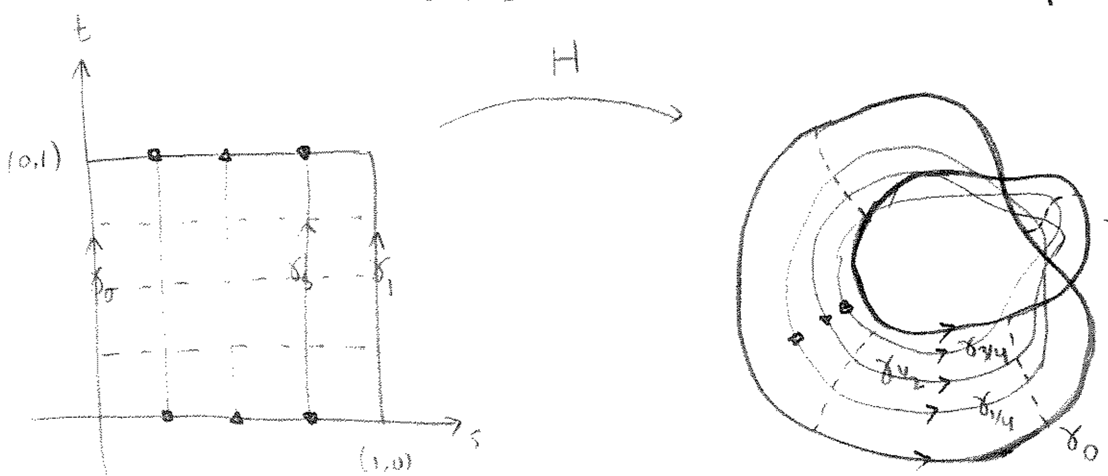
Then γ_0, γ_1 <u>are homotopic in A as closed curves</u>

if $\exists$ continuous homotopy $H : [0,1] \times [0,1] \to A$

s.t.

$$H(s,0) = H(s,1) \quad 0 \le s \le 1$$

, i.e. each $\gamma_s(t) = H(s,t)$ is closed. (but the terminal = initial endpoint may change w.r.t. s)

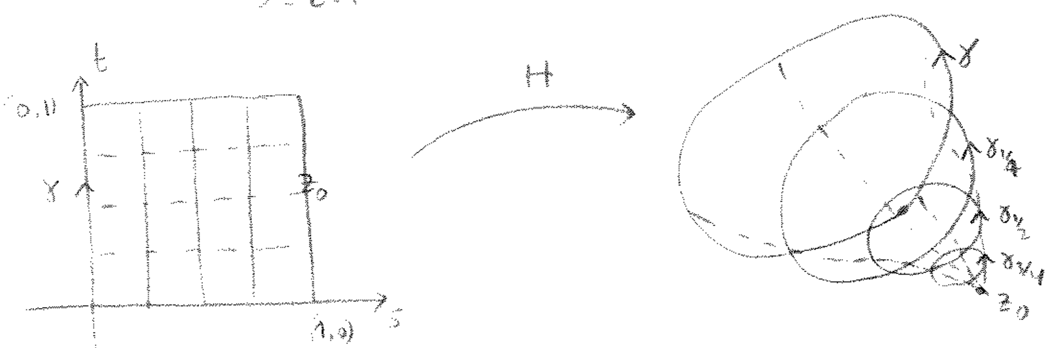


(label segments by their image contour)

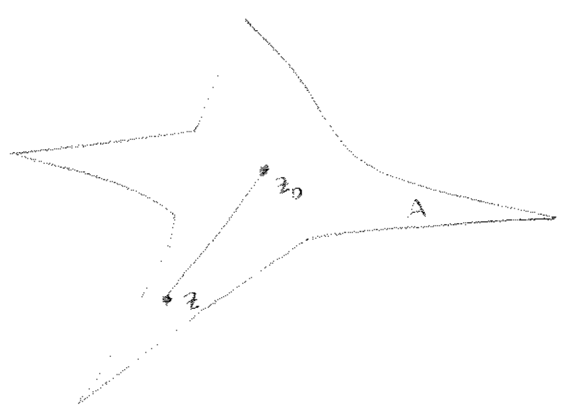
Def A connected open set A is <u>simply connected</u> iff every closed curve $\gamma : [0,1] \to A$ is homotopic as a closed curve to some point $z_0 \in A$, i.e.

$\exists\, H : [0,1]\times[0,1] \to A$ continuous

$$H(0,t) = \gamma(t) \qquad H(s,0) = H(s,1)$$
$$H(1,t) = z_0 \qquad 0 \le s \le 1$$
$$0 \le t \le 1$$

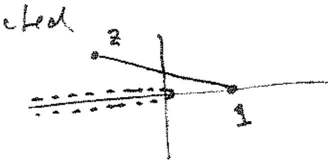


<u>Example</u> A is called <u>starshaped</u> iff $\exists\, z_0 \in A$ s.t. $\forall z \in A$, $0 \le s \le 1$
$$(1-s)z + s z_0 \in A$$

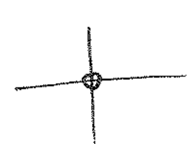


If A is starshaped and $\gamma : [0,1] \to A$ define
$$H(s,t) = (1-s)\gamma(t) + s z_0 \, ;$$
shrinks γ to z_0 as $0 \le s \le 1$.
So starshaped domains are simply connected.

<u>Example</u> : The domain for the standard branch of $\log z$, i.e. $\mathbb{C} \setminus \{x \mid x \in \mathbb{R}, x \le 0\}$ is star-shaped with respect to 1, hence simply connected.



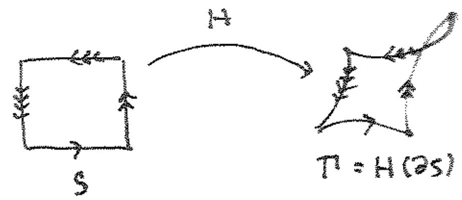
<u>Example</u> $\mathbb{C} \setminus \{0\}$ is <u>not</u> simply connected.
You prove this in HW using complex analysis, in fact using the theorems we prove today!



Homotopy Lemma : Let A open, connected, $f: A \to \mathbb{C}$ analytic

Let $S = \{(s,t) \mid 0 \le s \le 1,\ 0 \le t \le 1\}$ denote the unit square.
∂S the boundary, oriented c.c

Let $H: S \to A$ continuous
$\Gamma := H(\partial S)$ a p.w. C^1 contour.

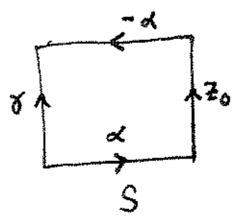


Then $\displaystyle\int_\Gamma f(z)\,dz = 0$

We will prove this lemma on the next page. It has the following consequences

Theorem 1 Antiderivative in simply connected domains:
If A is simply connected, $f: A \to \mathbb{C}$ analytic
then $\exists$ antideriv $F: A \to \mathbb{C}$, $F' = f$

pf: It suffices to prove $\int_\gamma f(z)\,dz = 0$ whenever γ is p.w. C^1 & closed, in A.
(because this implies path independence $\Rightarrow F\ \exists$.
For such γ consider a homotopy of γ to a fixed pt, thru closed curves



$$0 = \int_\Gamma f(z)\,dz = \cancelto{0}{\int_{z_0} f(z)\,dz} - \cancel{\int_\alpha f(z)\,dz} - \int_\gamma f(z)\,dz + \cancel{\int_\alpha f(z)\,dz} \qquad \blacksquare$$

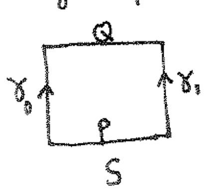
(technical pt: α might only be continuous, not p.w. C^1.
if so, look at proof of homotopy lemma on next page to see how to make proof rigorous.).

Theorem 2 Deformation Theorem
Let A open, connected (not nec. simply connected
$f: A \to \mathbb{C}$ analytic.

If γ_0, γ_1 are p.w. C^1 and homotopic in A either with fixed endpoints, or as closed curves

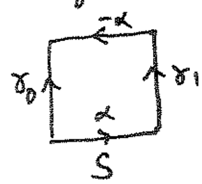
then $\displaystyle\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$

pf: $\gamma_0 \sim \gamma_1$ with fixed endpoints:



$$\Rightarrow \int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + 0 - \int_{\gamma_0} f(z)\,dz + 0 \qquad \blacksquare$$

$\gamma_0 \sim \gamma_1$ as closed curves:

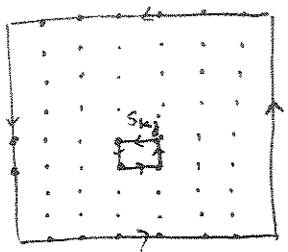


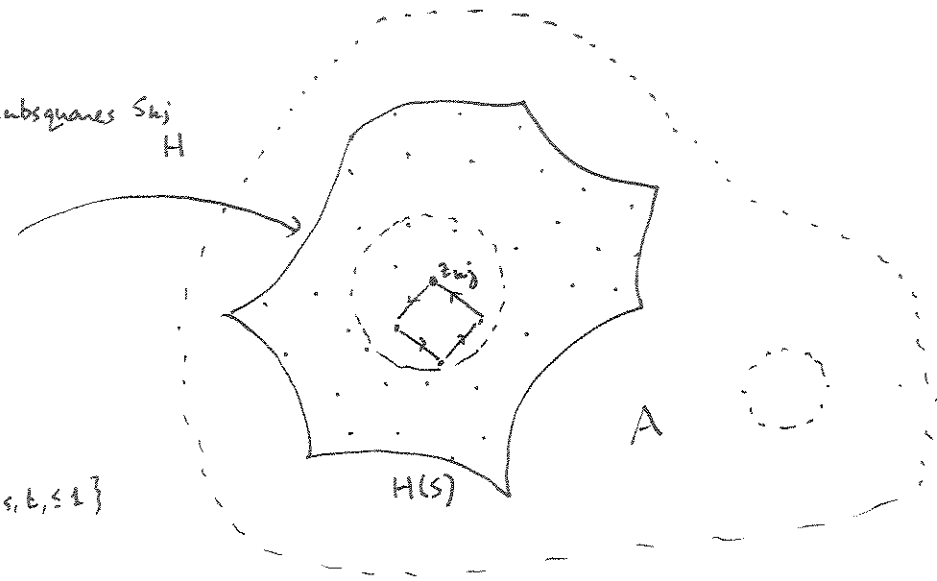
$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_\alpha f(z)\,dz - \int_{\gamma_0} f(z)\,dz + \int_\alpha f(z)\,dz. \qquad \blacksquare$$

Proof of homotopy lemma

subdivide S into n^2 subsquares S_{kj} of side-length $1/n$

H





$S = \{(s,t),\ 0 \le s, t, \le 1\}$

$H(\partial S) = \Gamma$

$H(\partial S_{kj}) := \Gamma_{kj}$

(replace any arcs of Γ_{kj} which are not p.w. C^1 with constant speed line segment paths between the images of vertices.)

By interior cancelation,

$$\int_\Gamma f(z)\,dz = \sum_{k,j=1}^{n} \int_{\Gamma_{kj}} f(z)\,dz$$

Note

1• $H(S)$ is compact, $\subset A$ (open) $\Rightarrow \exists\ \varepsilon > 0$ s.t. $D(z;\varepsilon) \subset A\ \forall z \in H(S)$
(positive distance lemma, §1.4)

2• H is continuous on S, hence uniformly continuous.
Thus for ε as in (1), $\exists\ \delta$ s.t. $\|(s,t)-(s',t')\| < \delta \Rightarrow |H(s,t) - H(s',t')| < \varepsilon$

3• If $\sqrt{2}/n < \delta$ as in (2)

then each $H(S_{kj}) \subset D(z_{kj};\varepsilon) \subset A \quad z_{kj} = H(s_k, t_j)$

4• By local antideriv. thm in $D(z_{kj};\varepsilon)$, $\int_{\Gamma_{kj}} f(z)\,dz = 0$

Thus, $\int_\Gamma f(z)\,dz = 0$ ∎