Math 4200
Fri 9/23

HW for Fri 9/30
2.2 8, 11
2.3 1, 3, 4, 5, 6, 7, 9, 10



- Discuss Green's Thm (Wed p. 4)
- Examples (Wed. p. 2-3)

We have carefully proven

- If $A$ is a connected open subset of $\mathbb{C}$, $f: A \to \mathbb{C}$ continuous.
  Then contour integrals $\int_\gamma f(z)\,dz$ are path independent
  if and only if $\exists$ antiderivative $F(z)$ to $f(z)$.

We have <u>less carefully</u> proven

- If $A$ is open and <u>simply connected</u>, $f: A \to \mathbb{C}$ analytic and <u>$C^1$</u>
  then $\int_\gamma f(z)\,dz$ are path independent, so $\exists$ antiderivative $F(z)$

<u>issues</u>:
- did not define simply connected precisely
- did not show path independence for <u>all</u> p.w. $C^1$ paths
  (assumed they crossed in such a was as to create a finite # of subdomains on which to apply Green)
- had to assume $f \in C^1$ (for Green's), which turns out to not be necessary.

<u>The goal of §2.3</u> is to deal with these issues,
and to also prove precise (and stronger) versions of the <u>deformation theorem</u>, about when you can change the contour curves of $\int_\gamma f(z)\,dz$ without changing the actual value of the contour integral.
Along the way we introduce the notion of <u>homotopy</u>, key to many areas of mathematics, especially <u>algebraic topology</u>

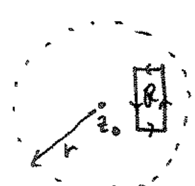
<u>Key step</u>: <u>Local antiderivative theorem</u>

Let $f: D(z_0; r) \to \mathbb{C}$ be analytic
Then $\exists\, F: D(z_0; r) \to \mathbb{C}$ s.t. $F' = f$ in $D$.

This local antiderivative theorem will follow from

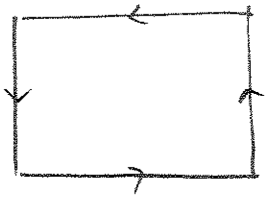
<u>Rectangle Lemma</u>: Let $f$, $D(z_0; r) = D$ as above.
Let $R = [a,b] \times [c,d] \subset D$, i.e. $R = \{x + iy, \mid a \le x \le b, c \le y \le d\}$
$\gamma = \partial R$ (counterclockwise)

Then $\oint_\gamma f(z)\,dz = 0$




(If $f$ was $C^1$ in $D$ this <u>would</u> follow from Green's Thm.)

Rectangle lemma proof: (Goursat)



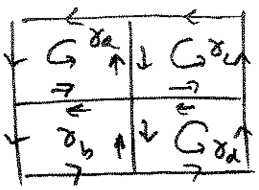
$R$, diagonal length $:= D$

perimeter $:= P$

$\gamma$ = p.w. $C^1$ bdry curve, consisting of 4 paths

want $\oint_\gamma f(z)\,dz = 0$

subdivide.

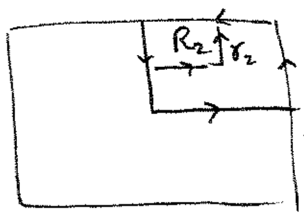



$$\int_\gamma f(z)\,dz = \int_{\gamma_a} + \int_{\gamma_b} + \int_{\gamma_c} + \int_{\gamma_d}$$

$$\Rightarrow \left|\int_\gamma f(z)\,dz\right| \leq \left|\int\right| + \left|\int\right| + \left|\int\right| + \left|\int\right|$$

$$\leq 4\left|\int_{\gamma_1} f(z)\,dz\right|$$

where $\left|\int_{\gamma_1} f(z)\,dz\right|$ is the max of the 4 moduli

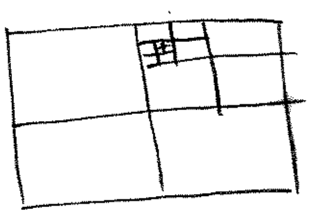



$\gamma_1 = \partial R_1$

pick $\gamma_2 = \partial R_2$ s.t.

$$\left|\int_{\gamma_1} f(z)\,dz\right| \leq 4\left|\int_{\gamma_2} f(z)\,dz\right|$$

Induct: $R \supset R_1 \supset R_2 \supset \dots \supset R_k$



$$\left|\int_\gamma f(z)\,dz\right| \leq 4^k \left|\int_{\gamma_k} f(z)\,dz\right|$$

$D_k$ = diagonal $(R_k) = 2^{-k} D$

$P_k$ = perimeter $(R_k) = 2^{-k} P$

Since the $\{R_k\}$ are nested, with diameter $\to 0$

$$\bigcap_{k=1}^{\infty} cl(R_k) = z_0 \in cl(R)$$

closure

punchline:

$f$ is analytic at $z_0$

thus

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + (z-z_0)\varepsilon(z) \qquad \varepsilon(z) \to 0 \text{ as } z \to z_0.$$

Let $\varepsilon > 0$. Pick $k$ s.t. $|\varepsilon(z)| < \varepsilon \ \forall z \in \bar{R}_k$

$$\Rightarrow \int_{\gamma_k} f(z)\,dz = \underbrace{\int_{\gamma_k} f(z_0) + f'(z_0)(z-z_0)\,dz}_{= f(z_0)z + f'(z_0)\left(\frac{z-z_0}{2}\right)^2\Big]_Q^Q \;=\; 0} + \int_{\gamma_k} (z-z_0)\varepsilon(z)\,dz$$

FTC!!, closed path

$$\Rightarrow \int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} (z-z_0)\varepsilon(z)\,dz$$

$$\Rightarrow |\eta_0| \le \int_{\gamma_k} \underbrace{|z-z_0|}_{\le D_k}\,\underbrace{|\varepsilon(z)|}_{\le \varepsilon}\,\underbrace{|dz|}_{\text{arclength}} \le \varepsilon D_k P_k = \varepsilon 4^{-k} DP$$

$$\Rightarrow \left|\int_{\gamma} f(z)\,dz\right| \le 4^k(\varepsilon 4^{-k} DP) = \varepsilon DP$$

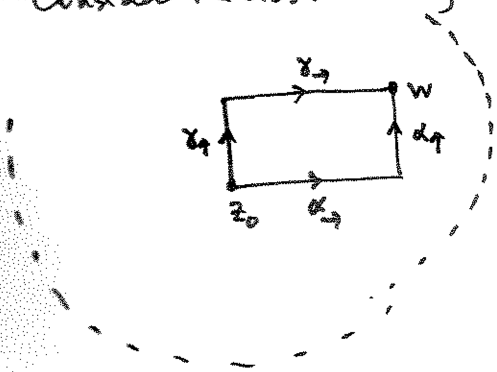
true $\forall \varepsilon > 0$

$$\Rightarrow \left|\int_{\gamma} f(z)\,dz\right| = 0$$ ∎, for Rectangle lemma.

$\left(\text{i.e. } \int_{\gamma} f(z)\,dz = 0\right)$

now complete the proof of local antiderivative theorem...
it's a modification of the Wed. antideriv. thm, but using only coord-dir. contours:

Let $w \in D(z_0; r)$.
Consider the closed rectangle $R(w)$ which has $z_0$ and $w$ as opposite corners
(this rect. may be a line segment, if $w - z_0$ is purely real or imag.)




Note $R(w) \subset D(z_0; r)$

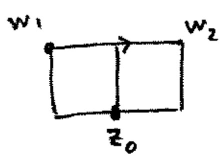
<u>Define</u> $F(w) := \int_{\gamma_{\rightarrow} + \alpha_{\uparrow}} f(z)\,dz$ (move horiz, then vert, from $z_0$ to $w$, around $\partial R(w)$)

$= \int_{\gamma_{\uparrow} + \gamma_{\rightarrow}} f(z)\,dz$ (vert, then horiz.) by Rect. lemma.

<u>Consequence:</u> If $w_1, w_2$ differ by a pure real, or pure imaginary number then

$$F(w_2) - F(w_1) = \int_{w_1 \rightarrow w_2} f(z)\,dz \quad \text{(horizontal contour)}$$

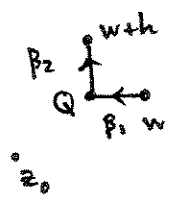
$$\text{or} = \int_{w_1 \uparrow w_2} f(z)\,dz \quad \text{(vertical contour)}$$




Finally, for $w \in D(z_0; r)$, $|h| < r - |w - z_0|$
consider

$F(w+h) - F(w) =$

$F(w+h) - F(Q)$
$+ F(Q) - F(w)$




$Q := \mathrm{Re}(w+h) + i\,\mathrm{Im}(w)$

$= \int_{\beta_1 + \beta_2} f(z)\,dz$, by <u>consequence</u>

Thus $F(w+h) - F(w) = \int_{\beta_1+\beta_2} f(z)\,dz = \int_{\beta_1+\beta_2} f(w)\,dz + \int_{\beta_1+\beta_2} f(z) - f(w)\,dz$

$= f(w)\,z\Big]_w^{w+h} + \text{error}$

$F(w+h) - F(w) = f(w)\,h + \text{error}$

$|\text{error}| \le \int_{\beta_1+\beta_2} |f(z) - f(w)|\,|dz| \le \underbrace{2|h|}_{\text{upper bd for curve length}} \underbrace{\max_{|z-w| \le 2|h|} |f(z) - f(w)|}_{\text{upper bound for integrand}}$

thus $\frac{|\text{error}|}{|h|} \to 0$ as $h \to 0$.

thus $F'(w) = f(w)$ ∎