Math 4200
Wed 9/21

- Finish examples on page 3 Monday notes
- Then discuss Theorem's 1 & 2 from Monday.
  They are restated in today's notes, with room for the proofs

Theorem 1 $\underset{\mathbb{C}}{\overset{A}{\cap}}$ open, connected, $f: A \to \mathbb{C}$ continuous.

Then the following are equivalent:

(i) $\exists\, F: A \to \mathbb{C}$ s.t. $F'(z) = f(z)\ \forall z \in A$ (and F is unique up to a constant).

(ii) $\forall$ choices of initial pt $P \in A$ and terminal pt $Q \in A$

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$ whenever γ_0, γ_1 are p.w. C^1, start at P and end at Q.
("path independence")

(iii) $\forall$ p.w. C^1 γ with are closed (i.e. initial pt = terminal pt),
$$\int_\gamma f(z)dz = 0$$

Steps: (i) $\Rightarrow$ (ii) $\Longleftrightarrow$ (iii):

then (ii) $\Rightarrow$ (i)
Pick $z_0 \in A$.
Define $F(w) = \displaystyle\int_{\gamma_{z_0,w}} f(z)dz$ where $\gamma_{z_0,w}$ is any p.w. C^1 path connecting z_0 to w

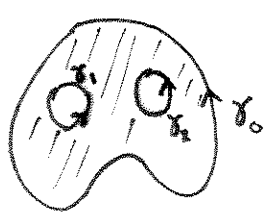
(ii) $\Rightarrow F(w)$ is well defined.
Show $F'(w) = f(w)$!

<u>Theorem 2</u> If A is <u>open</u> and <u>simply connected</u>, $f: A \to \mathbb{C}$ analytic and C^1
then $\exists\, F: A \to \mathbb{C}$ s.t. $F'(z) = f(z)\ \forall z \in A$

pf: Green's Thm! (to verify (iii) in <u>Theorem 1</u>)

Note, while proving Theorem 2, and appealing ~~th~~ to Green's Theorem (for domains with holes), we actually proved:

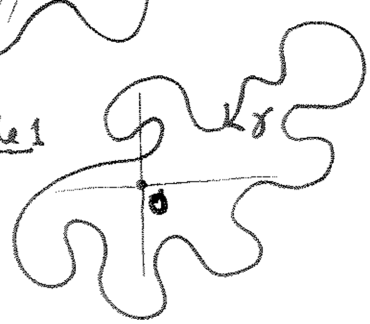
<u>Theorem</u> Let $f(z)$ be analytic and C^1 on an open region containing a compact subset A, whose boundary is a union of piecewise C^1 curves. Let γ_0 be the outer boundary curve, with $\gamma_1, \ldots$ the inner bdry curves. Orient curves as indicated



Then $\displaystyle\int_{\gamma_0} f(z)\,dz = \sum_{i \geq 1} \int_{\gamma_i} f(z)\,dz$

(notice we changed the orientation of the γ_i from Greens)

<u>Example 1</u>



$\displaystyle\int_{\gamma} \frac{1}{z}\,dz =$

<u>Example 2</u> Let $\gamma = \{z \mid |z| = 2\}$. (Oriented c.c.)

What is
$$\int_\gamma \frac{1}{z^3 - z}\, dz \ ?$$

note, $\dfrac{1}{z^3-z} = -\dfrac{1}{z} + \dfrac{1/2}{z-1} + \dfrac{1/2}{z+1}$

<u>Lemma</u>: Let γ be a circle centered at $\vec{z} = \vec{a}$

($i.e.$ $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$)

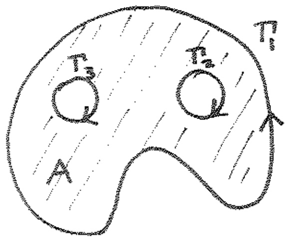
then $$\oint_{|z-a|=r} \frac{1}{z-a}\, dz = 2\pi i$$

http://en.wikipedia.org/wiki/Stokes'_theorem

Green's Theorem

(This is just one of the vector calculus "FTC"'s, and in fact one can understand all of them as special cases of a general theorem) called Stokes' Theorem

Let $\langle P(x,y), Q(x,y)\rangle$ be a vector field, C^1 on an open domain containing the set A and its boundary ∂A.
Orient $\Gamma = \partial A$ so that A is "on the left" as you traverse Γ:



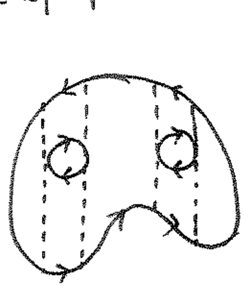
then
$$\boxed{\oint_{\partial A} P\,dx + Q\,dy = \iint_A (Q_x - P_y)\,dA}$$

proof: ① $\displaystyle\oint_{\partial A} P\,dx = \iint_A -P_y\,dA$, ② $\displaystyle\oint_{\partial A} Q\,dy = \iint_A Q_x\,dA$

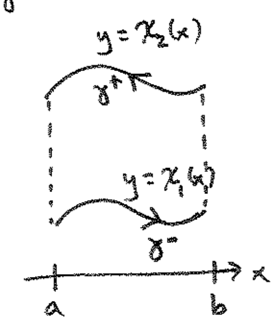
(unfortunately, I have used A for the region, and dA for dxdy — these uses of "A" are entirely independent)

① + ② = Green's Thm

① Chop up A into "vertical simple" subregions:



each region:

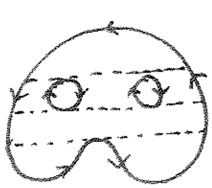


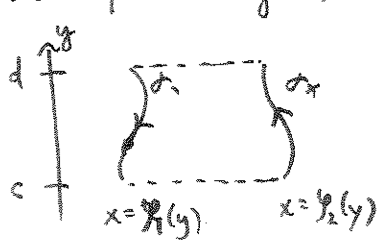
add these identities for each region, and use the fact that double integrals and line integrals are additive with respect to partitioning.
Deduce $\displaystyle\iint_A -P_y\,dA = \oint_{\partial A} P\,dx$ ■

iterate the area integral:

$$\int_a^b \int_{\chi_1(x)}^{\chi_2(x)} -P_y(x,y)\,dy\,dx$$
$$= \int_a^b -P(x,y)\Big]_{y=\chi_1(x)}^{y=\chi_2(x)} dx$$
$$= \int_a^b -P(x,\chi_2(x))\,dx + P(x,\chi_1(x))\,dx$$
$$= \int_{\gamma^+} P\,dx + \int_{\gamma^-} P\,dx$$

② Chop A into "horizontal simple" subregions





$$\int_c^d \int_{\varphi_1(y)}^{\varphi_2(y)} Q_x\,dx\,dy = \int_c^d Q(x,y)\Big]_{x=\varphi_1(y)}^{\varphi_2(y)} dy$$
$$= \int_c^d Q(\varphi_2(y),y)\,dy - Q(\varphi_1(y),y)\,dy$$

then add, as in ①. $\displaystyle = \int Q\,dy - \int Q\,dy$